Automorphism Groups of Trees: Prescribed Local Actions

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This talk split into two parts. First, the second author introduced basic properties of the universal group construction by Burger and Mozes, see Section 3.2 of [2]. Second, the first author described some variations of this construction by Banks, Elder and Willis (see [1]) and explained how this construction can be used to find infinitely many locally compact compactly generated non-discrete simple subgroups of tree automorphisms.

Universal Groups. Let $T_d = (X, Y)$ denote the d-regular tree ($d \geq 3$) and let $l : Y \to \{1, \ldots, d\}$ be a legal labelling of T_d. We adopt Serre's conventions for graph theory, see [4]. Given a vertex $x \in X$, every automorphism $g \in \mathrm{Aut}(T_d)$ induces a permutation at x given by $c(g, x) := l|_{E(gx)} \circ g|_{E(x)} \circ l|_{E(x)}^{-1} \in S_d$, where $E(x) := \{y \in Y \mid o(y) = x\}$.

Definition 1. Let $F \leq S_d$. Define $\mathrm{U}(F) := \{g \in \mathrm{Aut}(T_d) \mid \forall x \in X : \; c(g, x) \in F\}$.

The following proposition collects several basic properties of $\mathrm{U}(F)$. Also, it exemplifies the principle that properties of $\mathrm{U}(F)$ should correspond to properties of the finite permutation group F, which is part of the beauty of the construction.

Proposition 2. *Let $F \leq S_d$. Then the following statements hold.*

(i) $\mathrm{U}(F)$ *is closed in* $\mathrm{Aut}(T_d)$.
(ii) $\mathrm{U}(F)$ *is locally permutation isomorphic to* F.
(iii) $\mathrm{U}(F)$ *is vertex-transitive.*
(iv) $\mathrm{U}(F)$ *is edge-transitive if and only if* F *is transitive.*
(v) *Given legal labellings* l *and* l' *of* T_d, *the groups* $\mathrm{U}_{(l)}(F)$ *and* $\mathrm{U}_{(l')}(F)$ *are conjugate in* $\mathrm{Aut}(T_d)$.

Furthermore, it is immediate from Definition 1 that $\mathrm{U}(F)$ satisfies Tits' Independence Property. More precisely, we have the following.

Proposition 3. *Let $F \leq S_d$. Then $\mathrm{U}(F)^+$ is either trivial or simple. If F is transitive and generated by its point stabilizers, then $\mathrm{U}(F)^+ = \mathrm{U}(F) \cap \mathrm{Aut}(T_d)^+$ and hence $\mathrm{U}(F)^+ \leq \mathrm{U}(F)$ is of index two.*

Here, $\mathrm{U}(F)^+ := \langle \{g \in \mathrm{U}(F) \mid \exists y \in Y : \;\; gy = y\} \rangle$ is the subgroup of $\mathrm{U}(F)$ generated by edge-stabilizers. It is edge-transitive if and only if F is transitive and generated by its point stabilizers.

Finally, the term "universal" is justified by the following result.

Proposition 4. *Let $G \leq \mathrm{Aut}(T_d)$ be vertex-transitive and locally permutation isomorphic to a transitive permutation group $F \leq S_d$. Then there is a legal labelling l of T_d such that $G \leq \mathrm{U}_{(l)}(F)$.*

Universal groups have come up in the theory of lattices in products of two trees, see [3], but constitute interesting objects of study in themselves, too.

k-closures and Property P_k. Let T denote an infinite and locally finite tree (not necessarily regular) and $B(x,n)$ the ball of radius n centred at vertex x of T.

Definition 5. Let $G \leq \mathrm{Aut}(T)$ and $k \in \mathbb{N}$. The *k-closure* of G is

$$G^{(k)} := \{h \in \mathrm{Aut}(T) \mid \forall x \in X : \exists g \in G : h|_{B(x,k)} = g|_{B(x,k)}\}.$$

That is, the automorphisms of T that agree on each ball of radius k with some element of G.

In this setting, G is the analogue of F in the definition of $\mathrm{U}(F)$, providing a list of "allowed" actions. Notice also that $G^{(k)}$ is in some sense a "thicker" version of $\mathrm{U}(F)$ in that it has a prescribed local action on bigger balls (when $k > 1$).

Proposition 6. *The k-closure of G has the following basic properties.*

(i) *$G^{(k)}$ is a closed subgroup of* $\mathrm{Aut}(T)$.
(ii) *For every $k, l \in \mathbb{N}$ with $l < k$ we have $G \leq G^{(l)} \leq G^{(k)}$.*
(iii) $\bigcap_{k\in\mathbb{N}} G^{(k)} = \overline{G}$ *(the topological closure of G in* $\mathrm{Aut}(T)$*).*

Just as $\mathrm{U}(F)$ satisfies Tits' Independence Property (or Property P), the k-closure of G satisfies a "thicker" version of this property.

Definition 7. For any finite or (bi-)infinite path C in T and any $n \in \mathbb{N}$ let C^n be the subtree of T spanned by all vertices at distance at most n from C.

Let $G \leq \mathrm{Aut}(T)$, $k \in \mathbb{N}$ and C be a finite or infinite path in T. Then, for each vertex x of C, the point-wise stabilizer $\mathrm{Fix}_G(C^{k-1})$ of C^{k-1} in G acts on the "subtree rooted at x" (the subtree of T whose vertices are closer to x than to any other vertex of C) and we denote by F_x the permutation group induced by this action. We therefore have a map $\Phi : \mathrm{Fix}_G(C^{k-1}) \to \prod_{x\in C} F_x$ which is clearly an injective homomorphism.

We say that G satisfies *Property P_k* if for every finite or infinite path C the map Φ is an isomorphism.

Notice that when $k = 1$ we recover the original Property P defined by Tits ([5]).

Proposition 8. *Let $G \leq \mathrm{Aut}(T)$ and $k \in \mathbb{N}$, then $G^{(k)}$ satisfies Property P_k.*

It is almost immediate that this holds when C is an edge, whence it can easily be extended to finite paths. That it holds for (bi-)infinite paths follows from a limiting argument and the fact that $G^{(k)}$ is a closed subgroup of $\mathrm{Aut}(T)$.

Satisfying Property P_k characterizes when the process of taking k-closures stabilizes.

Theorem 9. *The group $G \leq \mathrm{Aut}(T)$ satisfies Property P_k for some k if and only if $G^{(k)} = \overline{G}$.*

More importantly, we deduce the following which will be used when finding infinitely many distinct simple subgroups.

Corollary 10. *There are infinitely many distinct k-closures of G if and only if $\overline{G}$ does not satisfy Property P_k for any k.*

To find simple subgroups we will use an analogous result to Tits' theorem ([5, Théorème 4.5]), with a similar proof. Let $G^{+_k} := \langle \mathrm{Fix}_G(e^{k-1}) \mid e \in Y \rangle$ denote the subgroup of G generated by pointwise stabilizers of "$(k-1)$-thick" edges.

Theorem 11. *Suppose $G \leq \mathrm{Aut}(T)$ does not stabilize a proper non-empty subtree or an end of T, and satisfies Property P_k. Then G^{+_k} is simple (or trivial).*

We have the following recipe to find simple subgroups of Aut(T): start off with some $G \leq \mathrm{Aut}(T)$ which does not stabilize a proper subtree of T, form its k-closures (they all satisfy Property P_k), use Theorem 11 to obtain the simple subgroups $(G^{(k)})^{+_k}$. We still need to ensure that the latter subgroups are non-discrete and different from each other, which will follow from the results below.

Lemma 12. *If $G \leq \mathrm{Aut}(T)$ does not stabilize a proper subtree of T we have*

(i) *$(G^{(k)})^{+_k}$ is an open subgroup of $G^{(k)}$.*
(ii) *$(G^{(k)})^{+_k}$ is non-discrete if and only if $G^{(k)}$ is non-discrete.*
(iii) *$(G^{(k)})^{+_k}$ satisfies Property P_k.*

Theorem 13. *Suppose that $G \leq \mathrm{Aut}(T)$ does not stabilize a proper subtree of T. Then $(G^{(r)})^{+_r} \leq (G^{(k)})^{+_k}$ for every $r \geq k$, with equality if and only if $G^{(r)} = G^{(k)}$.*

Thus, in order to construct infinitely many distinct t.d.l.c. simple non-discrete subgroups of $\mathrm{Aut}(T)$ it suffices to find examples with infinitely many distinct k-closures. By Corollary 10, this amounts to finding examples which do not satisfy Property P_k for any k.

Example 14. The following groups do not satisfy Property P_k for any k.

(i) $\mathrm{PSL}(2, \mathbb{Q}_p)$ acting on its Bruhat–Tits tree (which is isomorphic to T_{p+1}).
(ii) $\mathrm{BS}(m,n) := \langle a, t \mid t^{-1}a^m t = a^n \rangle$ (Baumslag–Solitar group) for coprime m, n acting on its Bass–Serre tree (which is isomorphic to T_{m+n}).

We note that this method finds infinitely many t.d.l.c. simple non-discrete groups which are pairwise distinct as subgroups of $\mathrm{Aut}(T)$. It would be desirable to know whether these subgroups are pairwise non-isomorphic. This is stated as work in progress in [1]. Using different methods, Simon Smith has found uncountably many t.d.l.c. simple non-discrete groups which are pairwise non-isomorphic. This was discussed in the talk by C. Reid and G. Willis.

References

[1] C. Banks and M. Elder and G. A. Willis, *Simple groups of automorphisms of trees determined by their actions on finite subtrees*, `http://arxiv.org/abs/1312.2311v2`.

[2] M. Burger and S. Mozes, *Groups acting on trees: from local to global structure*, Publications Mathématiques de l'IHÉS **92** (2000), 113–150.

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[4] J.-P. Serre, *Trees*, Springer Monographs in Mathematics.

[5] J. Tits, *Sur le groupe d'automorphismes d'un arbre*, in Essays on Topology and Related Topics, Springer Berlin Heidelberg (1970), 188–211.