On a generalization of Burger-Mozes universal groups (Focus on group theory day, 60 minutes, Newcastle)

27.07.15

Reminder (Burger-Mozes groups)

$T_d = (X, Y)$ d-regular tree ($d \geq 3$)

$l : Y \to \{1, \dots, d\}$ a legal labelling of T_d, i.e. for all $x \in X$: $l_x := l|_{E(x)}$ is a bijection and for all $e \in Y$: $l(e) = l(\bar{e})$.

We obtain a map $c : \mathrm{Aut}(T_d) \times X \to S_d$ by $(\alpha, x) \mapsto l_{\alpha x} \circ \alpha \circ l_x^{-1}$ satisfying $c(\alpha\beta, x) = c(\alpha, \beta x)\, c(\beta, x)$ (cocycle). This implies that the following subset of $\mathrm{Aut}(T_d)$ is a group.

Def. Let $F \leq S_d$. Set $U^{(l)}(F) := \{\alpha \in \mathrm{Aut}(T_d) \mid \forall x \in X : c(\alpha, x) \in F\}$.

We now generalize this construction by prescribing the local action on balls of a fixed radius $k \geq 1$. To this end, in addition to the above fix a "legally labelled" tree $B_{d,k}$, isomorphic to a ball of radius k in T_d and consider

$c_k : \mathrm{Aut}(T_d) \times X \to \mathrm{Aut}(B_{d,k})$, $(\alpha, x) \mapsto l^k_{\alpha x} \circ \alpha \circ l_x^{k^{-1}}$

where $l_x^k : B(x, k) \to B_{d,k}$ is the unique label-respecting isomorphism. This suggests:

Def. Let $F \leq \mathrm{Aut}(B_{d,k})$. Set $U_k^{(l)}(F) := \{\alpha \in \mathrm{Aut}(T_d) \mid \forall x \in X : c_k(\alpha, x) \in F\}$.

These groups share many basic properties with the Burger-Mozes groups:

Prop. Let $F \leq \mathrm{Aut}(B_{d,k})$. Then:

(i) $U_k^{(l)}(F) \leq \mathrm{Aut}(T_d)$ is closed

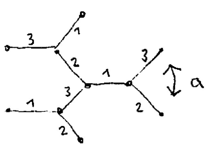
(ii) $U_k^{(l)}(F) \leq \mathrm{Aut}(T_d)$ is vertex-transitive

(iii) $U_k^{(l)}(F)$ satisfies Property P_k (Banks-Elder-Willis)

(iv) Given legal labellings $l, l' : Y \to \underline{d}$, the groups $U_k^{(l)}(F)$, $U_k^{(l')}(F)$ are conjugate in $\mathrm{Aut}(T_d)$.

(v) $U_k^{(l)}(F)$ is comp. gen., tot. disc., loc. compact Hausdorff with the subspace top.

The first curiosity arises when considering the local action. Whereas $U_1^{(l)}(F)$ locally realizes all elements of F, the same need not hold for $k \geq 2$. For instance, consider the two-element subgroup F of $\mathrm{Aut}(B_{3,2})$ generated by the element a:



Then there is no element in F which is compatible with a in direction 1.

We define:

Def. Let $F \leq \mathrm{Aut}(B_{d,k})$. Then F satisfies condition (C) if and only if ($U_k^{(l)}(F)$ is locally action isomorphic to F.) (i.e. $\forall x \in X$: $F = l_x^k \circ U_k^{(l)}(F)(x) \circ l_x^{k^{-1}}$).

The compatibility condition (C) can be phrased solely in terms of $F \le \mathrm{Aut}(B_{d,k})$: For $i \in \underline{d}$, let T_i be the subtree of $B_{d,k}$ given by the $(k-1)$-neighbourhood of the edge (b, b_i) where b is the center of $B_{d,k}$ and the label of (b, b_i) is i. Furthermore, let $\sigma_i : T_i \to T_i$ be the unique, non-trivial, label-respecting involution of T_i. Then:

$$\text{(C)} \quad \forall a \in F \ \forall i \in \underline{d} : \exists a_i \in F : \ a_i|_{T_i} = \sigma_{a(i)} \circ a \circ \sigma_i$$

("a_i is compatible with a in direction i")

Suppose that $F \le \mathrm{Aut}(B_{d,k})$ has (C). Then $U_k^{(1)}(F)$ is <u>discrete</u> if and only if

$$\text{(CR)} \quad \forall a \in F \ \forall i \in \underline{d} : \exists! \, a_i \in F : \ a_i|_{T_i} = \sigma_{a(i)} \circ a \circ \sigma_i$$

↑ "regular" ↑ uniqueness

Both conditions allow themselves for computations and need only be checked on generators. As an illustration we construct some <u>examples</u> for $k=2$. To this end, we view $\mathrm{Aut}(B_{d,2})$ as the following subgroup of $S_d \ltimes \prod_{i=1}^d S_d$:

$$\mathrm{Aut}(B_{d,2}) = \{ (a, (a_1, \dots, a_d)) \mid \forall i \in \underline{d} : a(i) = a_i(i) \} \le S_d \ltimes \prod_{i=1}^d S_d =: G_{d,2} \qquad S_d \wr S_d$$

To describe condition (C) and (CR) in this setting, we define

$\sigma_i : G_{d,2} \to G_{d,2}$, "swap a and a_i"

$p_i : G_{d,2} \to S_d \times S_d$, $(a, (a_1, \dots, a_d)) \mapsto (a, a_i)$

Then

$$\text{(C)} \quad \forall i \in \underline{d} : \ p_i F = p_i \sigma_i F$$

$$\text{(CR)} \quad \text{(C)} \ \& \ p_i^{-1}(\mathrm{id}, \mathrm{id}) = \mathrm{id}$$

Now define $\gamma : S_d \to \mathrm{Aut}(B_{d,2})$, $a \mapsto (a, (a, \dots, a))$. Then

$$\Gamma(F) := \mathrm{im}\, \gamma|_F$$

has (CR). All subgroups $F_2 \le \mathrm{Aut}(B_{d,2})$ which satisfy (C), project onto F and contain $\Gamma(F)$ are described as follows.

<u>Prop.</u> Let $F \le S_d$. Given $K \le \prod_{i=1}^d F(i) \cong \ker \pi \le \mathrm{Aut}(B_{d,2})$, there is $F_2 \le \mathrm{Aut}(B_{d,2})$ with (C) and fitting into $1 \to K \xrightarrow{\iota} F_2 \underset{\gamma}{\overset{\pi}{\rightleftarrows}} F \to 1$ if and only if K is invariant under $F \curvearrowright \prod_{i=1}^d F(i)$, $a \cdot (a_1, \dots, a_d) := (a a_{a^{-1}(1)} a^{-1}, \dots, a a_{a^{-1}(d)} a^{-1})$.

<u>Ex.</u> Given $N \trianglelefteq F(1)$ and $(f_i)_i$ in F with $f_i(1) = i$, set

$$\Delta(F, N) := \{ (a, (a f_1 a_1 f_1^{-1}, \dots, a f_d a_1 f_d^{-1})) \mid a \in F, \ a_1 \in N \} \qquad \text{(CR)}$$

$$\Phi(F, N) := \{ (a, (a f_1 a_1^{(1)} f_1^{-1}, \dots, a f_d a_1^{(d)} f_d^{-1})) \mid a \in F, \ a_1^{(i)} \in N \ \forall i \in \underline{d} \} \qquad \text{(C)}$$

As for the Burger-Mozes groups, a <u>universality</u> statement holds:

<u>Prop.</u> Let $H \le Aut(T_d)$ be vertex-tran., loc. tran. and contain an inv. edge-inversion. Then there is a legal lab. l of T_d such that

$$\begin{array}{ccccccccccc} U_1^{(l)}(F_1) & \ge & U_2^{(l)}(F_2) & \ge \dots \ge & U_k^{(l)}(F_k) & \ge & \dots & \ge & H & \ge & U_1^{(l)}(\{e\}) \\ \| & & \| & & \| & & & & \| & & \\ H^{(1)} & \ge & H^{(2)} & \ge \dots \ge & H^{(k)} & \ge & \dots & \ge & H & & \end{array}$$

where $F_k \le Aut(B_{d,k})$ is an action isom. to the action of H on k-balls

<u>Cor.</u> Let $H \le Aut(T_d)$ be discrete of order k, vertex-tran., loc. tran. and contain an inv. edge-inversion. Then $H = U_k^{(l)}(F_k)$ for some $F_k \le Aut(B_{d,k})$ w/ (CR).
$\pi_{k-1}F_k$ w/o (CR)

ord(H) = min{...

This leads to the following weakening of the <u>Goldschmidt-Sims conjecture</u>:

<u>Conj.</u> (G.-S.) Let T be a loc. fin. tree. Then there are only fin. many conj. classes of discrete, loc. prim. subgroups of $Aut(T)$.

<u>Conj.</u> (Weak G.-S.) Let T_d be the d-regular tree. Then there are only fin. many conj. classes of discrete, loc. prim., vertex-tran. subgroups of $Aut(T_d)$ containing an inv. edge-inversion.

We rephrase the 2nd conjecture in terms of the following:

<u>Def.</u> Let $F \le S_d$. Define

$$\text{cr-dim}(F) := \max\left\{ k \;\middle|\; \exists F_k \le Aut(B_{d,k}) : \begin{cases} \pi F_k = F \\ F_k \text{ has (CR)} \\ \pi_{k-1}F_k \text{ has not (CR)} \end{cases} \right\}$$

if the maximum exists and infinity otherwise.

$\text{cr-dim}(F) = \max\{\mathrm{ord}\, H \mid H \le Aut(T_d)\}$ w/ ...

$\mathrm{ord}\, H = \min\{k \mid H(B(1,k)) = \{id\}\}$

<u>Then:</u> Weak G.-S. $\Longleftrightarrow$ all primitive groups have finite cr-dimension.

<u>So far results</u> about this dimension:

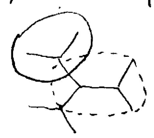
<u>Prop.</u> Let $F \le S_d$ be tran. Then $\text{cr-dim}(F) = 1 \Longleftrightarrow F$ regular.

<u>Prop.</u> Let $F \le S_d$ be prim., non-regular. If $F(1)$ has triv. nilpotent radical then $\text{cr-dim}(F) = 2$.

<u>Prop.</u> Let $F \le S_d$ and $P \le S_{d'}$ be tran. Then $\text{cr-dim}(F \wr P) \ge 3$.

<u>Prop.</u> (1980). We have $\text{cr-dim}(S_3) = 3$.

subnormal in $F(1)$
p-group



This includes:

(i) A_n, S_n $(n \ge 6)$ (AS)
(ii) Prim. groups of type (TW) (pt. stab. have triv. solv. rad.)
(iii) Prim. groups of simple diagonal type (ii) (HS) (pt. stab. have simple non-ab. socle)