On a generalization of Burger-Mozes universal groups 07/16

(Creswick, 30 minutes)

- $T_d = (X, Y)$ $(d \geq 3)$ d-regular tree
- $l : Y \longrightarrow \{1, \dots, d\}$ legal labelling, i.e. $\forall x \in X : l_x := l|_{E(x)}$ is a bijection and $\forall e \in Y : l(e) = l(\bar{e})$.

Instead of prescribing the local action on the 1-balls around vertices, we now prescribe it on k-balls around vertices for any given $k \geq 1$. To this end, fix a finite labelled tree isomorphic to a k-ball around a vertex in T_d. Consider

$$c_k : \mathrm{Aut}(T_d) \times X \longrightarrow \mathrm{Aut}(B_{d,k})$$
$$(\alpha, x) \longmapsto l^k_{\alpha x} \circ \alpha \circ (l^k_x)^{-1}$$

where $l^k_x : B(x,k) \longrightarrow B_{d,k}$ is the unique label-respecting isomorphism.

Def: Let $F \leq \mathrm{Aut}(B_{d,k})$. Set $U^{(l)}_k(F) := \{\alpha \in \mathrm{Aut}(T_d) \mid \forall x \in X : c_k(\alpha, x) \in F\}$.

Some of its properties follow as before:

Prop. Let $F \leq \mathrm{Aut}(B_{d,k})$. Then

(i) $U^{(l)}_k(F) \leq \mathrm{Aut}(T_d)$ is closed.

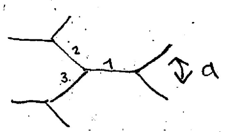
(ii) $U^{(l)}_k(F) \leq \mathrm{Aut}(T_d)$ is vertex-tran.

(iii) $U^{(l)}_k(F)$ satisfies property P_k (Banks-Elder-Willis) $\rightsquigarrow$ simple subgroups

(iv) Given legal labellings $l, l' : Y \to \{1, \dots, d\}$, the groups $U^{(l)}_k(F)$, $U^{(l')}_k(F)$ are conj. in $\mathrm{Aut}(T_d)$.

(v) $U^{(l)}_k(F)$ is comp. gen., tot. disc., loc. cpct., Hausdorff.

Contrasting the original case, $U_k(F)$ does not necessarily realize all of F locally. Consider



There is no element in F which is compatible with a in direction 1. (remember proof in $U(F)$-case)

Def. Let $F \leq \mathrm{Aut}(B_{d,k})$. Then F has (C) if $U_k(F)$ locally acts like F.

$(\iff \forall a \in F\ \forall i \in \{1, \dots, d\}\ \exists a_i \in F$: "a_i is compatible with a in direction i")

"direction"

What does this mean in terms of the action of F on $B_{d,k}$?

Consider

$$\mathrm{Aut}(B_{d,k}) \leq S_d \ltimes \prod_{i=1}^{d} S_d$$

as

$$\{(a,(a_1,\dots,a_d)) \mid \forall i\in\{1,\dots,d\}: a_i(i) = a(i)\}$$

Then (C) amounts to being able to "swap certain entries"...

Using this, we can produce some examples on the 2-sphere: Given $F \leq S_d$:

$$\Gamma(F) := \{(a,(a,\dots,a)) \mid a\in F\}$$

Then $U_2(\Gamma(F)) = D(F) = \{\alpha\in \mathrm{Aut}(T_d) \mid c_1(\alpha,x)\in F \text{ is constant over } X\}$

$\mathrm{Fix}_{\Gamma(F)} B(b,1) \cong \{e\}$.

$$\Phi(F) := \{(a,(a_1,\dots,a_d)) \mid a, a_i \in F\ ,\ a_i(i) = a(i)\}$$

Then $U_2(\Phi(F)) = U_1(F) = U(F)$ and $\mathrm{Fix}_{\Phi(F)} B(b,1) \cong F_1^d$.

If F is transitive, fix $(f_i)_{i=1}^{d}$ in F with $f_i(1) = i$. Set

$$\Delta(F) := \{(a,(af_1a_1f_1^{-1},\dots,af_da_1f_d^{-1})) \mid a\in F, a_i\in F_1\}$$

$\} f_{a(i)}a_1f_i^{-1}$

Then $U_2(\Delta(F)) \lneq U_1(F) = U(F)$ if F is non-regular, and

$\mathrm{Fix}_{\Delta(F)} B(b,1) \cong F_1$.

<u>Prop.</u> Let $F\leq S_d$ be 2-tran. and F_1 simple non-abelian. Suppose that $\tilde{F} \leq \mathrm{Aut}(B_{d,k})$ has (C) and $\pi\tilde{F} = F$. Then $U_k(\tilde{F})$ equals either

$$U_2(\Gamma(F)),\ U_2(\Delta(F)) \text{ or } U_2(\Phi(F)) = U_1(F).$$

<u>Proof</u> based on ideas of Burger-Mozes.

(If F_1 is not simple there are indeed for $N \trianglelefteq F_1$ groups

$\Phi(F,N) \leq \mathrm{Aut}(B_{d,2})$ with $\mathrm{Fix}_{\Phi(F,N)} B(b,1) \cong N^d$, and

$\Delta(F,N) \leq \mathrm{Aut}(B_{d,2})$ with $\mathrm{Fix}_{\Delta(F,N)} B(b,1) \cong N$.)

There is a universality statement for the $U_k(\tilde{F})$ which in fact characterizes them in the case where $\pi\tilde{F} = F$ is transitive. ↳

Prop. Let $H \leq \mathrm{Aut}(T_d)$ be vertex-transitive, locally transitive and contain an involutive edge-inversion. Then there is a legal labelling l of T_d s.t.

$$\begin{array}{ccccccccccc} U_1^{(l)}(F_1) & \geq & U_2^{(l)}(F_2) & \geq \cdots \geq & U_k^{(l)}(F_k) & \geq & \cdots \geq & H & \geq & U_1^{(l)}(\{e\}) \\ \| & & \| & & \| & & & \| & & \\ H^{(1)} & \geq & H^{(2)} & \geq \cdots \geq & H^{(k)} & \geq & \cdots \geq & H & & \end{array}$$

where $F_k \leq \mathrm{Aut}(B_{d,k})$ is action isomorphic to the action of balls on radius k.

Cor. The $U_k(\tilde{F})$ with $\pi\tilde{F}$ transitive are precisely the vertex-tran., loc. tran. subgroups of $\mathrm{Aut}(T_d)$ which contain an involutive edge-inversion and have P_k.

(using work of Banks - Elder - Willis)

Rem. Property P_k (for some k) is automatic if H is discrete. Thus studying discrete subgroups of the above kind reduces to studying $F \leq \mathrm{Aut}(B_{d,k})$ with a strengthening of (C) that amounts to discreteness of the associated universal group. This way, one can recover special cases of Weiss' conjecture, stating that there are only finitely many conjugacy classes in $\mathrm{Aut}(T_d)$ of vertex-tran., discrete groups for a given primitive local action.

E.g.: Primitive F s.t. F_1 has trivial nilpotent radical (using Thompson-Wielandt)

Potential application

Better understand projection closures of lattices $\Gamma \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2)$.

$H_i := \overline{\mathrm{pr}_i \Gamma} \neq \mathrm{Aut}(T_i)$ since top. fin. gen. (Burger - Mozes - Zimmer).