On Property (T) and amenability (Sydney, 24.10.16)

Both are properties of locally compact groups that can be viewed as dealing with unitary representations. Introduced by Kazhdan in 1967 (four pages) in the context of lattices of Lie groups, and von Neumann in 1929 in the context of the Banach-Tarski paradox respectively. Vital in rigidity theory of lattices in Lie groups and other fields.

[Recall that given a loc. cpct. group G, a unitary representation of G is a pair $(\pi, \mathcal{H})$ where $\pi : G \to \mathcal{U}(\mathcal{H})$ is a strongly continuous group homomorphism ($G \to \mathcal{H}$, $g \mapsto \pi(g)v$ is continuous $\forall v \in \mathcal{H}$). Here $\mathcal{U}(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$.]

__Def.__ (towards Property (T)) Let G be loc. cpct. and let $(\pi, \mathcal{H})$ be a unitary rep. of G. It almost has invariant vectors if for every compact $K \subseteq G$ and $\varepsilon > 0$ there is a unit vector $v \in \mathcal{H}$ s.t.

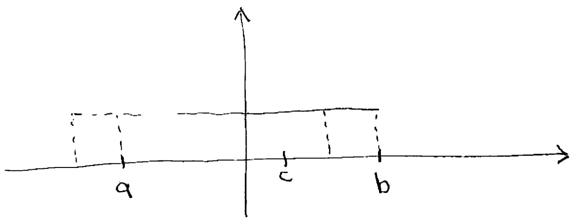
$$\sup_{g \in K} \| \pi(g)v - v \| < \varepsilon.$$

We say that v is (K, ε)-invariant.

__Ex.__ Let $G = \mathbb{R}$, $\mathcal{H} = L^2(\mathbb{R}, \mu_{Leb})$ and $\pi : G \to \mathcal{U}(\mathcal{H})$ the right-reg. rep.:

$$\pi : t \in \mathbb{R} \longmapsto (f \mapsto tf), \quad tf(x) = x + t \qquad \text{(left-shift)}$$

Here, one readily constructs (K, ε)-invariant vectors: Let $K \subseteq [-c, c]$ and choose $a < b \in \mathbb{R}$ s.t. c is small w.r.t. $b - a$, i.e. $c \ll b - a$. Then an (appropriately normalized) characteristic function on $[a, b]$ isn't translated off itself much by any $t \in K = [-c, c]$:



A similar argument works for $G = \mathbb{Z}$, $\mathcal{H} = L^2(\mathbb{Z}, \mu_{count})$.
However, in both cases there are no non-zero fully invariant vectors: this would imply essential constance of the vector. Being in $L^2(G, \mu)$ this would force μ to be finite, hence G to be compact.

<u>Def.</u> (Property (T)) A loc. cpct. group G has Property (T) if every unitary representation of G which almost has inv. vectors actually has a non-zero invariant vector.

By the example, $\mathbb{R}$ and $\mathbb{Z}$ don't have Property (T).

<u>Rem.</u> The notation (T) stands for the trivial representation T being isolated in the space of unitary representations of G (for a certain topology). Translating this yields the above def.

<u>Prop.</u> Let G be compact. Then G has Property (T).

<u>Proof.</u> Let $(\pi, \mathcal{H})$ be a unitary rep. of G which almost has invariant vectors. In particular there is a unit vector $v_0 \in \mathcal{H}$ which is $(G, \frac{1}{2})$-invariant: $\sup_{g\in G} \| \pi(g) v_0 - v_0 \| < \frac{1}{2}$. Define

$$v := \int_G \pi(g) v_0 \ \mu(g) \qquad \text{(Bochner integral)}$$

Then v is G-invariant by the properties of the integral and the rep.: For all $h \in G$ and $w \in \mathcal{H}$ we have

$$\langle \pi(h) v, w \rangle \overset{\text{unit.}}{=} \langle v, \pi(h)^{-1} w \rangle \overset{\text{int.}}{=} \int_{g\in G} \langle \pi(g) v_0, \pi(h)^{-1} w \rangle \ \mu(g)$$

$$= \int_G \langle \pi(hg) v_0, w \rangle \ \mu(g) \overset{\text{Haar}}{=} \int_G \langle \pi(g') v_0, w \rangle \ \mu(g') = \langle v, w \rangle .$$

Also, $\| v - v_0 \| \overset{\text{norm.}}{=} \| \int_G \pi(g) v_0 - v \ \mu(g) \| \leq \int_G \| \pi(g) v_0 - v \| \ \mu(g) < \frac{1}{2}$. □

<u>Inheritance properties</u> (unitary rep's at work)

<u>Prop.</u> Let G and H be loc. cpct. If G has Property (T) and $\varphi: G \to H$ is a morphism with dense image then H has Property (T).

<u>Proof</u> Let $(\pi, \mathcal{H})$ be a unitary rep. of H which almost has invariant vectors. Then $(\pi \circ \varphi, \mathcal{H})$ is a unitary rep. of G w/ the same property. Hence $(\pi\circ\varphi, \mathcal{H})$ has a non-zero invariant vector, i.e. $\varphi(G) \leq H$, and therefore H by continuity.

To deal with extensions, we say that a pair (G, H) of loc. cpct. groups $H \leq G$ has Property (T) if every unit. rep. of G which almost has inv. vectors, has a H-inv. vector.

Prop Let $1 \to G_1 \xrightarrow{i} G \xrightarrow{pr} G_2 \to 1$ be a short exact sequence of locally compact groups. Then G has Property (T) if and only if (G, G_1) and G_2 do.

Prop. If G has (T) then so does G_2 by the above. Also (G, G_1) has (T), in fact (G,G) has (T).

For the converse, let $(\pi, \mathcal{H})$ be a unitary rep. of G which almost has inv. vectors. Then $\mathcal{H}^{G_1}$ is non-trivial since (G, G_1) has (T) and is G-invariant since $G_1 \trianglelefteq G$. Thus $(\pi, \mathcal{H}^{G_1})$ is a unitary rep. of G as well. And it almost has invariant vectors, too (consider $P: \mathcal{H} \to \mathcal{H}^{G_1}$). Now, $(\pi, \mathcal{H}^{G_1})$ factors through G_2, which has Property (T). Hence there is a non-zero inv. vector. $\square$

More difficult facts:

Prop. Let G be loc. cpct. and second-countable. Further, let $\Gamma \leq G$ be a lattice. Then G has Property (T) $\iff$ Γ has Property (T).

Prop. A real simple Lie group of rank ≥ 2 has Property (T).

Ex. $SL(n, \mathbb{R})$, $n \geq 3$. (not out-of-this-world-difficult)

Kazhdan used Property (T) to show that certain groups are compactly (finitely) generated. (beautiful argument!)

Prop. Let G be a locally compact group with Property (T). Then G is compactly generated.

Proof. Let $\mathcal{G}$ be the collection of compactly generated open subgroups of G. By local compactness $G = \bigcup_{H \in \mathcal{G}} H$. Also, G/H is discrete for all $H \in \mathcal{G}$. Consider

$$(\pi, \mathcal{H}) := \left(\widehat{\bigoplus} \lambda_{G/H}, \widehat{\bigoplus} \ell^2(G/H) \right) \quad \text{(left-regular representation)}$$

This unit. rep. of G almost has invariant vectors. In fact, any compact $K \subseteq G$ is contained in some $H_K \in \mathcal{G}$ and hence $\delta_{eH_K} \in \mathcal{H}$ is K-inv. Thus $(\pi, \mathcal{H})$ has a non-zero inv. vector $v \in \mathcal{H}$. If the H-component of v is non-trivial, then G/H is finite and hence G is gen. by a cpct. gen. set for H and fin. many coset rep's. $\square$

App. G real simple Lie w/ rank ≥ 2. $K \leq G$ max. compact, $\Gamma \leq G$ torsion-free lattice. Then $\Gamma = \pi_1(\Gamma \backslash G / K)$ (loc. symm. space) is fin. gen.

Property (T) & amenability

Def. (Amenability) A loc. cpct. group G is amenable if the right-reg. rep. $(\rho_G, L^2(G,\mu))$ almost has inv. vectors.

It is a non-triv. fact that the above is equivalent to:

Def. (Amenability) A loc. cpct. group G is amenable if for every continuous action of G on a cpct. metric space X there is a G-inv. prop. measure on X.

Prop. Locally compact abelian groups are amenable by the second definition and Kakutani-Markov.

Prop. Compact groups are amenable by the second def. and averaging via the Pettis integral.

Prop. Dense images of amenable groups are amenable by the second definition.

Prop. Let G be locally compact amenable. Then G has Property (T) if and only if it is compact.

Proof. Since G is amenable, $(\rho_G, L^2(G,\mu))$ almost has inv. vectors. Then Property (T) implies that there is a non-zero invariant vector which forces G to be compact as before. Conversely...

Closed subgroups, lattices.

Consequences of Property (T)

Prop. Let G have (T). Any morphism of G into an amenable group has relatively compact image.

Proof. ...

Prop. Let G have Property (T). Then

(i) any morphism from G to $\mathbb{R}^m \oplus \mathbb{Z}^n$ $(m,n \geq 0)$ is trivial,

(ii) G is unimodular, and

(iii) $G/\overline{[G,G]}$ is compact.

Non-Ex.

(i) Free groups don't have Property (T).

(ii) $\pi_1(\Sigma_g) =: \Gamma$ $(g \geq 1)$ does not have (T).

$\ulcorner$ $\Gamma/\overline{[\Gamma,\Gamma]} = H_1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^{2g}$ is not finite. $\lrcorner$