p-localization of Burger–Mozes universal groups

(Group actions seminar, University of Sydney, 27.02.17)

Introduction: Structure theory of t.d.l.c. groups

$$\begin{array}{ccccc} G_0 & \rightarrowtail & G & \twoheadrightarrow & G/G_0 \\ \text{conn.} & & \text{loc.cpct.} & & \text{t.d.l.c.} \end{array}$$

Gleason, Yamabe, Montgomery, Zippin

many, new impetus due to Willis '94

Many new concepts being developed to understand general t.d.l.c. groups. E.g.: Is there a reduction scheme to locally (virtually) pro-p groups (for profinite groups, many results were known for pro-p groups first)

Reid '11 Let G be a t.d.l.c. group and p prime.

The p-localization $G_{(p)}$ of G is constructed as follows:

1. Pick a compact open subgroup $K \leq G$.
2. Pick a Sylow p-subgroup $S \leq K$.
3. Define $G_{(p)} := \mathrm{Comm}_G(S) \left(= \{ g \in G \mid [S : S \cap gSg^{-1}], [gSg^{-1} : S \cap gSg^{-1}] < \infty \} \right)$, equipped with the unique group top. s.t. $S \rightarrowtail \mathrm{Comm}_G(S)$ is open.

(Recall: $\mathrm{Comm}_G(S) = \{ \dots \} \geq N_G(S)$)

Thm. Let $G, p, K, S, G_{(p)}$ be as above.

(i) The map $G_{(p)} \rightarrowtail G$ is injective (immediate), continuous (easy) and has dense image. (hard)

(ii) The G-conjugacy class and top. isom. class of G don't depend on the choices of S and K. (hard)

Thm. (...) Let $x \in G_{(p)}$. Then

(i) $\Delta(x)_p = \Delta_{(p)}(x)$ (medium)

(ii) $s(x)_p \leq s_{(p)}(x)$ (can be made more precise) (hard)

Example: Burger–Mozes universal groups (Burger–Mozes '00)
(acting on regular trees, locally like a given permutation group)

$\boxed{\text{Who has heard about these?}}$

Let Ω be a set of cardinality $d \in \mathbb{N}_{\geq 3}$ and $F \leq \mathrm{Sym}(\Omega)$.
Denote by $T_d = (V, E)$ the d-regular tree and let $l : E \to \Omega$ be a legal labelling, i.e. $\forall x \in V$: $l_x : E(x) = \{ e \in E \mid o(e) = x \} \to \Omega$ is a bijection and $\forall e \in E$: $l(e) = l(\bar{e})$. Then consider

$$\sigma : \mathrm{Aut}(T_d) \times V \to \mathrm{Sym}(\Omega), \ (\alpha, x) \mapsto l_{\alpha x} \circ \alpha \circ l_x^{-1}$$

(local action ... )

<u>Def.</u> (Burger–Mozes '00) F, l as above.

$$U^{(l)}(F) := \{ \alpha \in \mathrm{Aut}(T_d) \mid \forall x \in V : \sigma(\alpha, x) \in F \}$$

(changing the labelling l amounts to passing to a conjugate of $U^{(l)}(F) \leq \mathrm{Aut}(T_d)$. $\rightsquigarrow$ fix l and omit)

Recall: $\mathrm{Aut}(T_d)$ is t.d.l.c. when equipped w/ the permutation topology for its action on the vertices of T_d.

<u>Prop.</u> $F \leq \mathrm{Sym}(\Omega)$. Then $U(F)$ is
(i) closed in $\mathrm{Aut}(T_d)$,
(ii) locally permutation-isomorphic to F,
(iii) vertex-transitive
(iv) edge-transitive if and only if F is transitive, and
(v) discrete in $\mathrm{Aut}(T_d)$ if and only if F is semiregular (free action)

Then $U(F)$ is a (compactly generated) t.d.l.c. group in its own right.

(Optional: Universality statement ...)

<u>Ex.</u> $U(\mathrm{Sym}(\Omega)) = \mathrm{Aut}(T_d)$, $U(\{e\}) \cong \ast_{i=1}^{d} \mathbb{Z}/2\mathbb{Z}$, $U(\text{semiregular } F)_b \cong F$.

p-localization

1. For every finite subtree T of T_d, the fixator $U(F)_T$ of T in $U(F)$ is compact open.

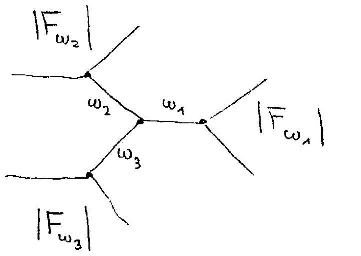
2. How to find a Sylow p-subgroup of $U(F)_T$?

A natural choice of p-subgroup of $U(F)_T$ is $U(F(p))_T$ where $F(p) \leq F$ is a Sylow p-subgroup. However, this need not be maximal. However, it is if taking point stabilizers in F and taking Sylow p-subgroups commute:

<u>Prop.</u> $F \leq \mathrm{Sym}(\Omega)$. Let $F(p) \leq F$ be a Sylow p-subgroup. Then $U(F(p))_T \leq U(F)_T$ is a Sylow p-subgroup if and only if $F(p)_\omega \leq F_\omega$ is for every $\omega \in \Omega$.

<u>Sketch</u>: This is a counting argument, based on the local structure of $U(F)$. For simplicity, assume $T = b \in V$ is a single vertex. Then

$$\left| U(F)_b \big|_{S(b,k)} \right| = \left| U(F)_b \big|_{S(b,k-1)} \right| \cdot \prod_{x \in S(b,k-1)} |F_{\omega_x}| .$$



(...)

<u>Examples / Non-Examples</u>

1. $F = S_3$, $p = 2$ does not work. E.g. $F(2) = \langle (12) \rangle$ and $F(2)_3 = \{id\}$ is not a Sylow 2-subgroup of $F_1 = \langle (23) \rangle$.
2. $F = \mathrm{Sym}(\Omega)$ or $F = \mathrm{Alt}(\Omega)$, $p^s \,|\, |\Omega|$ maximal. Then it works if and only if either
   - (i) $p > d$
   - (ii) $s \geq 1$ and $p^{s+1} > |\Omega|$, or (e.g. $F = S_6$, $p = 3$)
   - (iii) $F = \mathrm{Alt}(\Omega)$ and $(d,p) = (3,2)$

   (use iterated wreath product structure of $\mathrm{Sym}(\Omega)(p)$.)
3. If $\Omega / F = \Omega / F(p)$, it works. (E.g. F trans., $d = p^n$)

3. In the above case, what can be said about $\mathrm{Comm}_{U(F)}(U(F(p))_T)$?

Essential Lemma (Caprace-Monod '11, Caprace-Reid-Willis '13)

Let G be profinite and $K \leq G$ closed. Then

$$\mathrm{Comm}_G(K) = \bigcup_{L \leq_o K} N_G(L)$$

Sketch "$\supseteq$": immediate; "$\subseteq$": Let $g \in \mathrm{Comm}_G(K)$ and consider $H := \langle K, g \rangle$. Show that H inherits residual discreteness from G and hence has small invariant neighbourhoods, i.e. g normalizes some $L \leq_o K$.

We have the following (notation explained after)

Prop. (T. '17) Let $F \leq \mathrm{Sym}(\Omega)$ and $F(p) \leq F$ p-Sylow such that $F(p)_\omega \leq F_\omega$ is p-Sylow for all $\omega \in \Omega$. Put $S := U(F(p))_b$. Then

$$\mathrm{Comm}_{U(F)}(S) \geq \langle G(F(p), F), \{\Gamma_{V/L}(N_F(F(p))) \mid L \leq_o S\} \rangle$$

Notation For permutation groups $F \leq F' \leq \mathrm{Sym}(\Omega)$ define

$\to G(F) := \{\alpha \in \mathrm{Aut}(T_d) \mid \sigma(g, v) \in F$ for all but finitely many $x \in V\}$

(notice: $U(F) \leq G(F)$. Make $U(F)$ open $\rightsquigarrow$ unique grp. top.)

$\to G(F, F') := G(F) \cap U(F')$ (i.e. exceptions in F')

Let $\mathcal{P} = (P_i)_{i \in I}$ be a partition of V. Set

$\to \Gamma_{\mathcal{P}}(F) := \{\alpha \in \mathrm{Aut}(T_d) \mid \forall i \in I\ \exists \tau_i \in F : \forall x \in P_i : \sigma(\alpha, x) = \tau_i\}$

(constant local permutation in F on elements of $\mathcal{P}$)

Sketch. Show that $U(\{id\}) \leq \mathrm{Comm}_{U(F)}(S)$ (orbit-stabilizer-theorem) $\rightsquigarrow$ vertex-transitive. It then suffices to look at $\mathrm{Comm}_{U(F)_b}(S)$. (...)

Thm. (T. '17) Let $F \leq \mathrm{Sym}(\Omega)$ and $F(p) \leq F$ p-Sylow. If $\Omega / F = \Omega / F(p)$ and $N_{F_\omega}(F(p)_\omega) = F(p)_\omega$ $\forall \omega \in \Omega$ then $U(F)_{(p)} = G(F(p), F)$.

(generally: $G(F(p), F) \leq U(F)_{(p)} \leq U(F)$, all three cases occur)

Sketch: Use the lemma and contradict the assumption $N_{F_\omega}(F(p)_\omega) = F(p)_\omega$.

t.d.l.c. G

pick local Sylow $S \leq K$ compact open

$G_{(p)} := \mathrm{Comm}_G(S)$ w/ unique group top. s.t. $S \leq G_{(p)}$ open

- behaves well w.r.t. modular function & scale.
- $G_{(p)} \rightarrowtail G$ injective, continuously w/ $\boxed{\text{dense image}}$

$x \in G$, $S \leq K$ local p-Sylow

$\Longrightarrow$ $xSx^{-1} \leq xKx^{-1}$ local p-Sylow

Lem 3.2, part (iii)

$S \overset{p\text{-Syl.}}{\underset{\text{loc.}}{\leq}} G \rightsquigarrow S \overset{p\text{-Syl.}}{\leq} C$ compact open

talking about profinite groups only ?!

...

[$\mathrm{core}(U \leq G) \overset{?}{=} \bigcap_{g \in G} gUg^{-1}$ (normal core)]

$S \leq K \leq G$ loc. Sylow, i.e. $S \in \mathcal{L}(G)$

$\uparrow$ compact open

$x \in G$. Then $xSx^{-1} \in \mathcal{L}(G)$.

Find $u \in U \leq G$ open s.t. uSu^{-1} is commensurate to xSx^{-1}.

Then S is commensurate to $u^{-1}xSx^{-1}u$, i.e. $u^{-1}x \in \mathrm{Comm}_G(S)$,

i.e. $x \in U\,\mathrm{Comm}_G(S)$.

$\left[S \overset{p\text{-Syl.}}{\leq} G,\ H \leq G,\ K = \mathrm{Core}_G(H) \overset{?}{\Longrightarrow} K \cap S \overset{p\text{-Syl.}}{\leq} K \right]$

$K = \bigcap_{g \in G} gHg^{-1}$

$[G:S]_p = 1.$

$[K : K \cap S] = \mathrm{lcm}_{N \trianglelefteq_o K} \{ \underbrace{[K/N : (K \cap S)N/N]}_{\text{3rd}} \}$

$[K : (K \cap S)N]$

$\overset{?}{=}$ $\mathrm{lcm}_{N \trianglelefteq_o G} \{ [G/N : SN/N] \} = [G:S]$