Totally disconnected, locally compact groups from transcendental field extensions

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(20+5 minutes)

(joint work (in progress) with Timothy P. Bywaters)

In the structure theory of locally compact (l.c.) groups, totally disconnected (t.d.) ones play a crucial role due to the short exact sequence

$$
\begin{array}{ccccccccc}
1 & \to & G^\circ & \to & G & \to & G/G^\circ & \to & 1 \\
 & & \text{connected l.c.} & & \text{l.c.} & & \text{t.d.l.c.} & &
\end{array}
$$

and the fact that connected l.c. groups are inverse limits of Lie groups.

Field extensions have been rediscovered as a source of t.d.l.c. groups:
Let $K \subseteq E$ be a field extension. Consider $\mathrm{Aut}_K(E) = \{g \in \mathrm{Aut}(E) \mid \forall x \in K : gx = x\}$
For any set X, we equip $H \leq \mathrm{Sym}(X)$ with the permutation topology:
A basis of open sets is $\{U_{x,y} \mid x, y \in X^n\}$ where $U_{x,y} := \{g \in H \mid \forall i : gx_i = y_i\}$.

With this topology, $\mathrm{Aut}_K(E)$ is a t.d. Hausdorff group. When is it locally compact, non-discrete, compactly generated, simple, ...?

<u>Fact</u>: $H \leq \mathrm{Sym}(X)$ is compact if and only if it is closed and all its orbits are finite.

<u>Prop.</u> Let $K \subseteq E$ be a field extension. If $\mathrm{tr\text{-}deg}(E:K) < \infty$ then $\mathrm{Aut}_K(E)$ is locally compact. Conversely, if $\mathrm{Aut}_K(E)$ is locally compact, and E is algebraically closed, then $\mathrm{tr\text{-}deg}(E:K) < \infty$.

<u>Proof</u>: Let $M \subseteq E$ be a transcendence basis. If M is finite then $\mathrm{Aut}_K(E)_M = \mathrm{Aut}_{K(M)}(E)$ is open. It has finite orbits because $K(M) \subseteq E$ is an algebraic extension. Hence it is compact.

Conversely, if $\mathrm{Aut}_K(E)$ is locally compact, then it has an identity neighbourhood basis of compact open subgroups (van Dantzig). Hence $\mathrm{Aut}_K(E)_S$ is compact for some finite set $S \subseteq E$.
Suppose tr-deg$(E:K)$ is infinite. Then so is tr-deg$(E:K(S))$. Let $(X_i)_{i\in I}$ be a tr-basis of E over $K(S)$. Since E is alg. closed, it is an alg. closure of $K(S)((X_i)_{i\in I})$. Hence any automorphism of $K(S)((X_i)_{i\in I})$ extends to an automorphism of E. Thus $\mathrm{Aut}_K(E)_S$ has infinite orbits. Contradiction. □

Ex. $\boxed{1.}$ $K \subseteq K(X)$. Any automorphism in $\mathrm{Aut}_K(K(X))$ is determined by its image on X, and this image must be of the form

$$\frac{aX+b}{cX+d} \quad \text{for some} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,K).$$

Hence $\mathrm{Aut}_K(K(X)) \cong PGL(2,K)$ with the discrete topology.

Robert $\boxed{2.}$ Let K be a field and $n \in \mathbb{N}_{\geq 2}$. Consider the extension

$$K \subseteq K(X) \subseteq K(X)(X^{n^{-1}}) \subseteq K(X)(X^{n^{-1}}, X^{n^{-2}}) \subseteq \cdots$$

$$E := K(X)(\{X^{n^{-k}} \mid k \in \mathbb{N}\}) \quad \text{(non-finitely generated)}$$

For appropriate K, one finds $\mathrm{Aut}_K(E) \cong \mathbb{Z} \times C_2$ with the discrete top.