<u>Maths Club Presentation</u> (30 minutes (first part)) 30.04.19

From linear algebra to topology – the power of abstraction

It all starts with $\mathbb{R}^3$. We have vectors

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

and can compute <u>angles</u> between them

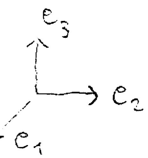
$$\langle v, w \rangle = \|v\| \, \|w\| \cdot \cos(\sphericalangle(v,w)) = v_1 w_1 + v_2 w_2 + v_3 w_3 = 0$$

(right angle)

where $\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ is the <u>length</u> of v.

Every vector can be split into its components:

$$w = \langle w, e_1 \rangle e_1 + \langle w, e_2 \rangle e_2 + \langle w, e_3 \rangle e_3$$



$$= -1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We also have the <u>distance</u> between the points v, w:

$$d(v,w) = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2 + (v_3 - w_3)^2}$$

All of this is formalized:

$V := \mathbb{R}^3$ is a <u>vector space</u>

(a set with an addition and scalar multiplication)

$\langle -, - \rangle : V \times V \to \mathbb{R}$ is an <u>inner product</u>

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$

(ii) $\langle x, y \rangle = \langle y, x \rangle$

(iii) $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$

If $(e_1, \ldots, e_n)$ is an <u>orthonormal</u> <u>basis</u> of V

$\langle e_i, e_j \rangle = 0$ $\langle e_i, e_i \rangle = 1$

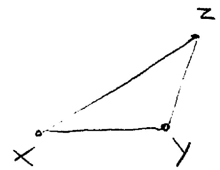
then for any $v \in V$: $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$.

The set $X := V$, together with the map $d : X \times X \to \mathbb{R}$ is a <u>metric space</u>:

(i) $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,z) = d(x,y) + d(y,z)$



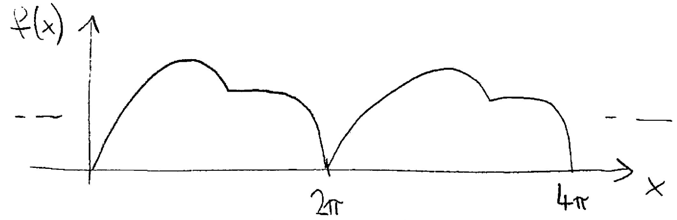
The theory of vector and metric spaces is often applied in unexpected settings.

! imprecise

① Fourier series

$$V := C_{2\pi}(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ continuous}, \ f(x+2\pi) = f(x) \ \forall x \}$$

E.g. $\sin(x), \cos(x) \in C_{2\pi}(\mathbb{R})$, vector space!



$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(t)\,\overline{g(t)}\, dt$$ is an inner product!

$(e_k)_{k \in \mathbb{Z}}$ where $e_k(x) = e^{ikx} = \cos(kx) + i\sin(kx)$ is an orthonormal basis.

So any vector $f \in V$ can be written as

$$f = \sum_{k \in \mathbb{Z}} \langle f, e_k \rangle e_k$$

Pictures : Wikipedia

② Dynamical Systems

About metric spaces (X, d) : A map $C : X \to X$ such that there is $0 \leq \lambda < 1$ so that $\forall x, y \in X$:

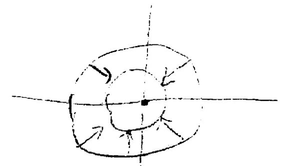
$$d(C(x), C(y)) \leq \lambda\, d(x, y)$$

is a contraction.

Theorem (Banach '22) Every contraction has a unique fixed point, i.e. a point $x \in X$ s.t. $C(x) = x$.

Example: $(X, d) = (\mathbb{R}^2, d)$.

$C : x \longmapsto \lambda x$. Unique fixed point $0 \in \mathbb{R}^2$:



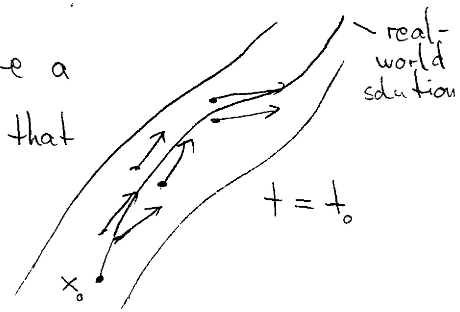
$F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ time-dependent vector field

Given $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, is there a function $x(t) : \mathbb{R} \to \mathbb{R}^n$ such that

(i) $x(t_0) = x_0$

(ii) $x'(t) = F(t, x)$?

"differential equation"



Theorem (Picard-Lindelöf

Yes, but only $x(t): \underbrace{(t_0-\delta, t_0+\delta)}_{I_\delta} \to \mathbb{R}^n$.

Idea

We need a function $x(t)$ with $x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))\,ds$

Indeed, then $x(t_0) = x_0$ and $x'(t) = F(t, x(t))$

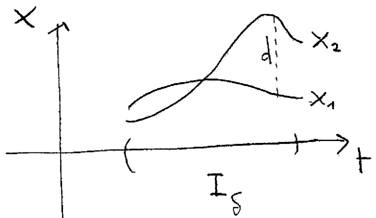
↑ fundamental thm. of calculus

Let $X = \{ x : I_\delta \to \mathbb{R}^n \}$ and

$$d(x_1, x_2) = \sup \{ \| x_1(t) - x_2(t) \| \mid t \in I_\delta \}$$

Define $C : X \to X$ by

$$C(x)(t) := x_0 + \int_{t_0}^{t} F(s, x(s))\,ds$$



Then we are looking for a fixed point of C:

$$x(t) = C(x)(t) = x_0 + \int_{t_0}^{t} F(s, x(s))\,ds.$$

Show that C is a contraction. Then Banach guarantees a solution!