Groups acting on trees with prescribed local action

(Prague, 05.06.19, 60+ minutes)

Why groups acting on trees?

Let G be a group. Group theory first distinguishes between finite and infinite groups.

1. Composition series:

   $$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

   with G_{i+1}/G_i simple

2. Jordan-Hölder:

   Uniqueness of subquotients

3. Classification of finite simple groups

$\rightsquigarrow$ fairly well understood

1. Adian-Rabin '55:

   Isomorphism theorem for finitely presented groups is undecidable.

2. Olshansky, Vaughan-Lee '70:

   There exist continuously many different varieties of groups (closed under homomorphic images, subgroups, cartesian products)

So we have to put some kind of restriction on the class of infinite groups we study.

Now, let G be a locally compact group. That is G carries a topology for which the group operations are continuous that is Hausdorff (points separated by open sets) and locally compact (every point has a compact neighbourhood).

This is actually not a restriction yet: Any abstract group is locally compact when equipped with the discrete topology.

$$1 \longrightarrow G^\circ \longrightarrow G \longrightarrow G/G^\circ \longrightarrow 1$$

connected component of the identity in G: closed and normal subgroup.

is an inverse limit of Lie groups (possibly 0-dim.)
(Hilbert's 5th problem; Gleason, Yamabe, Montgomery – Zippin; 50's)
$\rightsquigarrow$ fairly well understood

quotient is <u>totally disconnected</u> (every point is its own connected component) and locally compact

<u>Abels '73</u> (after Cayley, Schreier)
Let G be a t.d.l.c. group. Then G acts vertex-transitively on a connected, locally finite graph Γ with compact open vertex stabilisers if and only if G is compactly generated.
(finite generation in the discrete case)

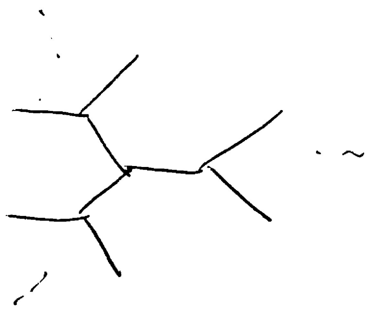
Let G be a compactly generated t.d.l.c. group.

<u>Examples</u>

1. $F_{\{a,b\}} \curvearrowright \mathrm{Cay}(F_{\{a,b\}}, \{a,b\})$

2. Compact t.d. group $\curvearrowright \bullet$

3. $\mathrm{Aut}(T_3) \curvearrowright T_3$



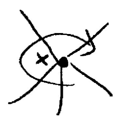
open neighbourhoods of identity are (pointwise) stabilisers of finite sets (permutation topology)

$$\begin{tikzcd}
 & & & 1 \arrow[d] & & \\
 & & & \pi_1(\Gamma) \arrow[d] & & \\
 & & & \widetilde{G/G^\circ} \arrow[dd] \arrow[dr] & & \\
 & & & & \mathrm{Aut}(T_d) \arrow[dd, dash, dashed] & \\
1 \arrow[r] & G^\circ \arrow[r] & G \arrow[r] & G/G^\circ \arrow[rr] \arrow[d] \arrow[dr] & & 1 \\
 & & & 1 & \mathrm{Aut}(\Gamma) &
\end{tikzcd}$$

Let $T_d = (V, E)$ be the d-regular tree ($d \in \mathbb{N}_{\geq 3}$) and let $H \leq \mathrm{Aut}(T_d)$. Given $x \in V$, the local action of H at x is the permutation group

$$H_x = \mathrm{Stab}_H(x) \curvearrowright E(x) := \{ e \in E \mid o(e) = x \}$$

Let Ω be a set. A permutation group $F \leq \mathrm{Sym}(\Omega)$ could be



2-transitive $\Rightarrow$ primitive $\Rightarrow$ quasiprimitive $\Rightarrow$ semiprimitive $\Rightarrow$ transitive

primitive: preserves no non-trivial partition

quasiprimitive: transitive & all non-trivial normal subgroups transitive

semiprimitive: transitive & all normal subgroups either transitive or semiregular

$A_5 \curvearrowright A_5/D_5$ $A_5 \curvearrowright A_5/C_5$ $C_4 \trianglerighteq C_2$ $D_4 \trianglerighteq C_2 \times C_2$

We are interested in results where the local action properties have a global impact on the group. Most prominently, there is the following structure theorem.

For any t.d.l.c. group H we define

$$H^{(\infty)} := \bigcap \{ N \trianglelefteq H \mid N \text{ closed normal cocompact} \}$$

$$= \bigcap \{ K \trianglelefteq H \mid K \leq H \text{ open, finite index} \}$$

think kernel of adjoint representation of Lie group

$$QZ(H) := \{ h \in H \mid C_H(h) \leq H \text{ is open} \} \quad \text{"quasi-center"}$$

<u>Thm.</u> (Burger–Mozes '00, T. '17) Let Γ be a loc. fin. conn. graph. Let $H \leq \mathrm{Aut}(\Gamma)$ be closed non-discrete, and $\boxed{\text{locally semiprimitive}}$. Then

i.e. every local action

(i) $H^{(\infty)}$ is minimal closed normal cocompact in H,

(ii) $QZ(H)$ is maximal discrete normal, and non-cocompact in H,

(iii) $H^{(\infty)}/QZ(H^{(\infty)}) = H^{(\infty)}/(QZ(H) \cap H^{(\infty)})$ admits minimal, non-trivial closed normal subgroups; finite in number, H-conjugate and top. simple.

(iv) If Γ is a tree and, in addition, H is loc. prim., then $H^{(\infty)}/QZ(H^{(\infty)})$ is a direct product of top. simple groups.

(This resembles semisimple Lie groups)

A new class of examples

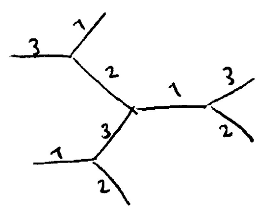
Constructing groups with a given local permutation group instead of obtaining permutations from a given group.

Let Ω be a set (of labels) of cardinality d and let l be a labelling of T_d, i.e. a map $l : E \to \Omega$ such that for every vertex $x \in V$ the map $l|_{E(x)} : E(x) \to \Omega$ is a bijection for every $x \in V$ (and $l(e) = l(\bar{e})$ for all $e \in E$). Further, fix a tree $B_{d,k}$ isomorphic to a labelled ball of radius k around a vertex in T_d.

$d = 3$

$\Omega = \{1,2,3\}$

$k = 2$



The map

$$\sigma_k : \mathrm{Aut}(T_d) \times V \to \mathrm{Aut}(B_{d,k})$$

$$(g, x) \longmapsto l_{gx} \circ g \circ l_x^{-1}$$

captures the <u>k-local action</u> of g at x

$(l_x : B(x,k) \to B_{d,k}$ unique label-preserving$)$

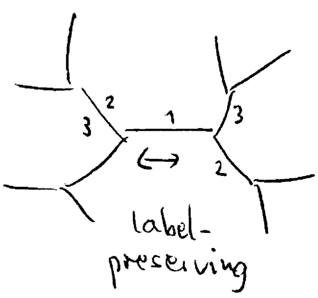
<u>Definition</u> (Burger–Mozes '00, T. '14) Let $F \leq \mathrm{Aut}(B_{d,k})$. Define

$$U_k^{(l)}(F) := \{ g \in \mathrm{Aut}(T_d) \mid \forall x \in V : \sigma_k(g,x) \in F \}$$

For example:

1) $U_k(\mathrm{Aut}(B_{d,k})) = \mathrm{Aut}(T_d)$

2) $U_1(\{\mathrm{id}\}) \cong \underbrace{\mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}}_{d}$



To get an idea how to deal with this kind of group:

<u>Prop.</u> Let $F \leq \mathrm{Aut}(B_{d,k})$. The group $U_k(F)$ is

(i) closed in $\mathrm{Aut}(B_{d,k})$

(ii) vertex-transitive

(iii) for $k=1$, the 1-local action of $U_k(F)$ is F, for $k \geq 2$, it may be smaller than F.

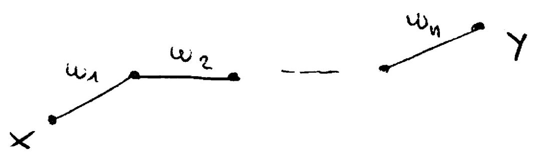
<u>Proof</u>

(i) We show that the complement is open: If $g \notin U_k(F)$ then there $x \in V$ such that $\sigma_k(g,x) \notin F$, and

$$\{ h \in \mathrm{Aut}(T_d) \mid h|_{B(x,k)} = g|_{B(x,k)} \}$$

is an open neighbourhood of g contained in the complement of $U_k(F)$ in $\mathrm{Aut}(T_d)$.

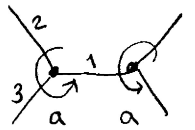
(ii) We have $U_k(F) \geq U_k(\{id\}) = U_1(\{id\})$ which is already vertex-transitive:



Let $\iota_w^{(x)}$ be the element in $U_1(\{id\})_x$ which inverts the edge issuing from x labelled w.

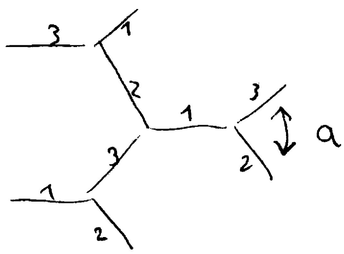
Then $\iota_{w_1}^{(x)} \circ \dots \circ \iota_{w_n}^{(x)}$ maps x to y, because each $\iota_w^{(x)}$ is label-preserving.

(iii) ~~Define~~ For $a \in F$ define $\alpha \in \mathrm{Aut}(T_d)$ by setting $\alpha(x) = x$ and $\sigma_1(\alpha, y) = a$ for all $y \in V$.



Note that this need not work when $k \geq 2$. Indeed, for

F:



$F = \{id, a\}$

we have $U_2(F) = U_2(\{id\})$ because the element a element a cannot be "extended".