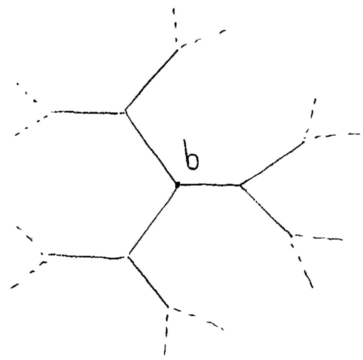
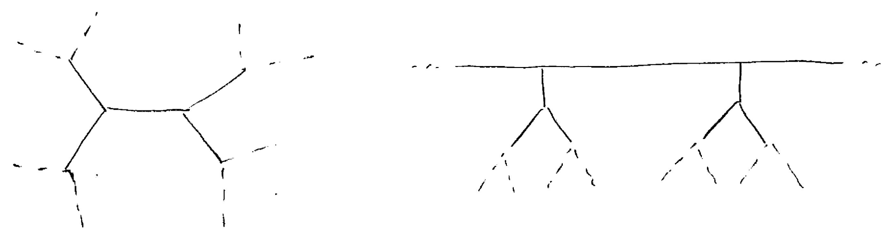
What CARMA Seminar, 12.09.19
60 minutes



Let $T_d = (V, E)$ be the d-regular tree and $\mathrm{Aut}(T_d)$ its group of symmetries.



The group $\mathrm{Aut}(T_d)$ contains "rotations", "inversions" and "translations".

It can be equipped with the topology for which a neighbourhood basis of $\mathrm{id} \in \mathrm{Aut}(T_d)$ is given by the sets

these are actually compact open subgroups

$$\mathrm{Stab}_{\mathrm{Aut}(T_d)}(B(b,n)) = \{ g \in \mathrm{Aut}(T_d) \mid g|_{B(b,n)} = \mathrm{id} \}, \quad n \in \mathbb{N}_0$$

not connected.

With this topology, $\mathrm{Aut}(T_d)$ becomes a totally disconnected (every element is its own connected component) locally compact (every element has a compact neighbourhood) topological group

[Examples: $(\mathbb{Q}, +)$]

Why (Slide presentation)

What exactly

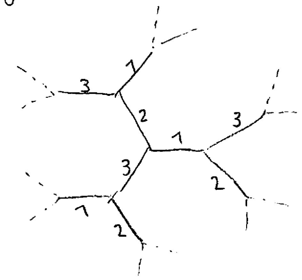
Is there a general structure theory for subgroups of $\mathrm{Aut}(T_d)$ like there is for other classes of groups?

- finite: (see slides)
- finitely generated abelian groups: $\mathbb{Z}^r \times \mathbb{Z}/q_1\mathbb{Z} \times \dots \times \mathbb{Z}/q_n\mathbb{Z}$
- Lie groups: Semisimple $\rtimes$ Solvable

Q. Given $H \leq \mathrm{Aut}(T_d)$ what are all the normal subgroups of H?

How

Label the edges E of T_d with labels $\{1, \dots, d\}$ in a regular fashion. Then to every $g \in \mathrm{Aut}(T_d)$ and $x \in V(T_d)$ there is an associated permutation $\sigma(g,x) \in S_d$:



$$\sigma : \mathrm{Aut}(T_d) \times V(T_d) \longrightarrow S_d$$
$$(g, x) \longmapsto \sigma(g,x)$$

"the permutation that g induces at $x \in V$"

For a fixed x, we have $\sigma_x(H) := \sigma(\mathrm{Stab}_H(x), x) \leq S_d$, the <u>local action</u> of H at x.

<u>Example / Definition</u> Let $F \leq S_d$. Define

$$U(F) := \{ g \in \mathrm{Aut}(T_d) \mid \sigma(g,x) \in F \text{ for all } x \in V \}.$$

The local action of $U(F)$ at any vertex is at most F, and in fact equal to F.

Properties of F correspond to properties of $U(F)$. For example,

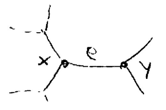
- $U(F)$ is discrete $\iff$ F is semiregluar (has trivial point stabilisers)

- $U(F)$ is edge-transitive $\iff$ F is transitive (has only one orbit)

<u>global property</u> <u>local property</u>

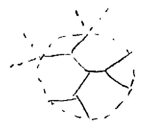
<u>Prop.</u> Let $F \leq S_d$ be semiregular. Then $U(F)$ is discrete.

<u>Proof.</u> Let $e \in E(T_d)$ be an edge. We show that $\mathrm{Stab}_{U(F)}(e) = \{id\}$. Let $g \in \mathrm{Stab}_{U(F)}(e)$



Since $\sigma_x(U(F))$ is semiregular, any $g \in \mathrm{Stab}_{U(F)}(e) \leq \mathrm{Stab}_{U(F)}(x)$ all edges issuing from x. Similarly for y. This propagates.

Conversely:



(not necessary)

global assumption

Prop. Let $H \leq \mathrm{Aut}(T_d)$ be $\boxed{\text{vertex-transitive}}$ and $\boxed{\text{locally semiprimitive}}$

local action $\quad C_4 \trianglerighteq C_2$

not tran.

global implication

small

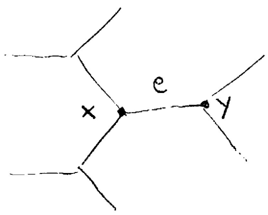
Then every normal subgroup of H is $\boxed{\text{either discrete or cocompact.}}$

big

Proof Let N be a normal subgroup of H.

Then $\forall x$: $\mathrm{Stab}_N(x) \trianglelefteq \mathrm{Stab}_H(x)$, so we have $\sigma_x(N) \trianglelefteq \sigma_x(H)$, which is semiprim.
So $\sigma_x(N)$ is either semiregular or transitive. Furthermore, since H is vertex-transitive and $N \trianglelefteq H$, the groups $\sigma_x(N)$ $(x \in V(T_d))$ are conjugate in S_d. In particular, if $\sigma_x(N)$ is semiregular ~~for~~ some x then it is for every $x \in V(T_d)$. Similarly, for transitivity.

If $\sigma_x(N)$ is semiregular (i.e. $N \curvearrowright \{1, \dots, d\}$ freely, i.e. trivial point point stabilisers) then N is discrete: Indeed, let $e \in E(T_d)$ be any edge, then $\mathrm{Stab}_N(e) = \{id\}$:



Let $g \in \mathrm{Stab}_N(e)$. Because $\sigma_x(N)$ is semiregular, g fixes all edges issuing from x.
Similarly, $g \in \mathrm{Stab}_N(y)$ and fixes therefore all edges issuing from y. This propagates.

On the other hand, if all $\sigma_x(N)$ are transitive then x can be mapped to any vertex at distance 2 (using the vertex-stabiliser of the vertex in between). $\searrow$ quotient $N \backslash T_d$ is , or

How