# Think globally, act locally

Stephan Tornier

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

December 9, 2020

# Summary

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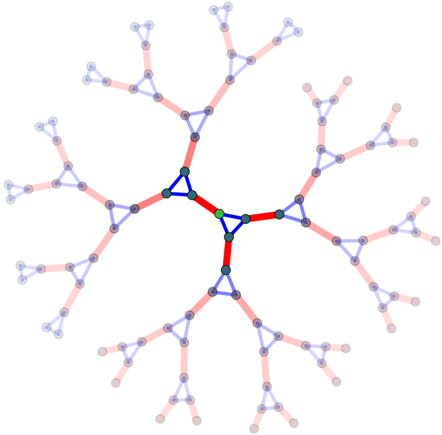

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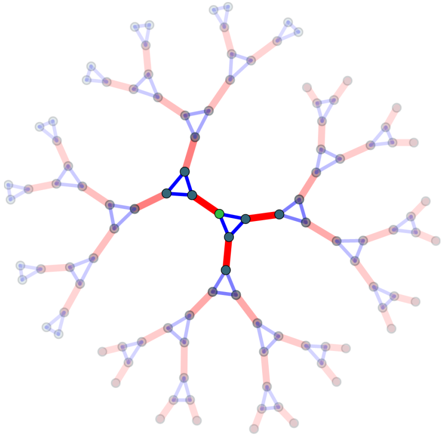

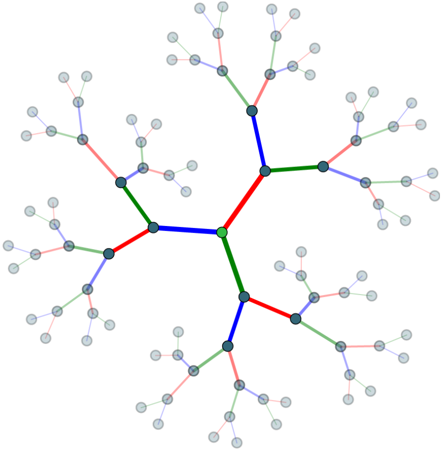

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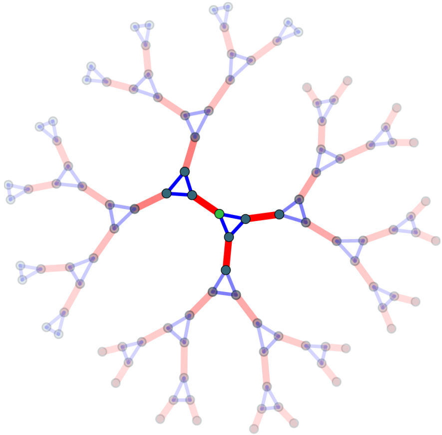

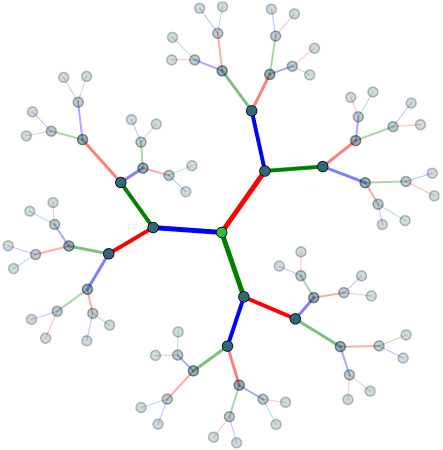

https://zerodimensional.group

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Ask me about it in gather.town!

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4. $H^{(\infty)}/\mathrm{QZ}(H^{(\infty)})$ *admits non-trivial, minimal closed normal subgroups; finitely many, $H$-conjugate and topologically simple.*

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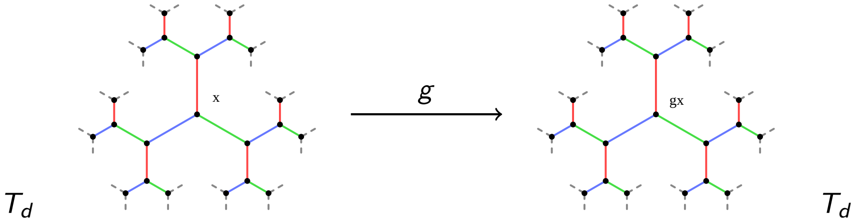

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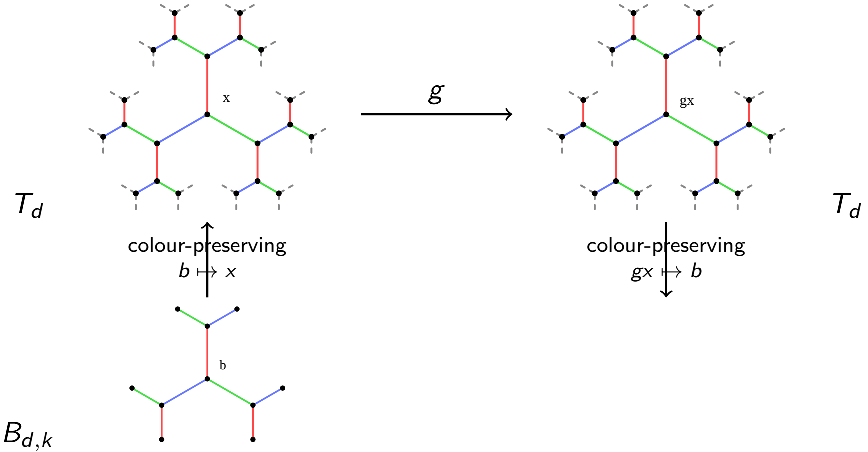

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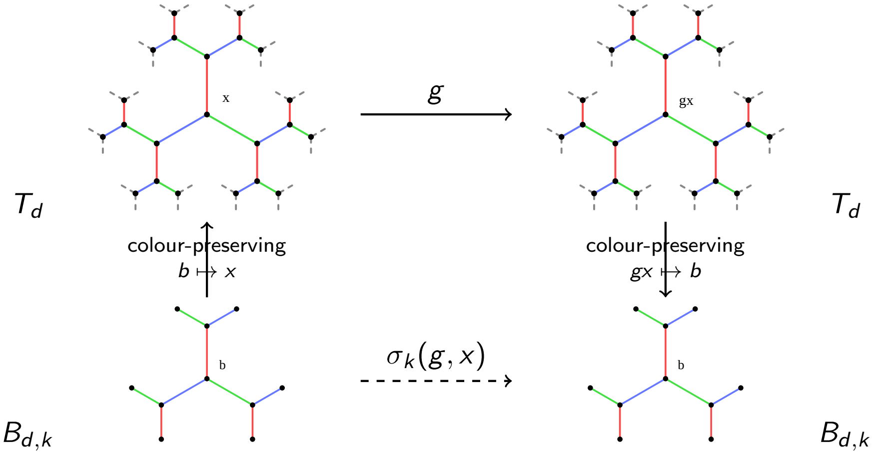

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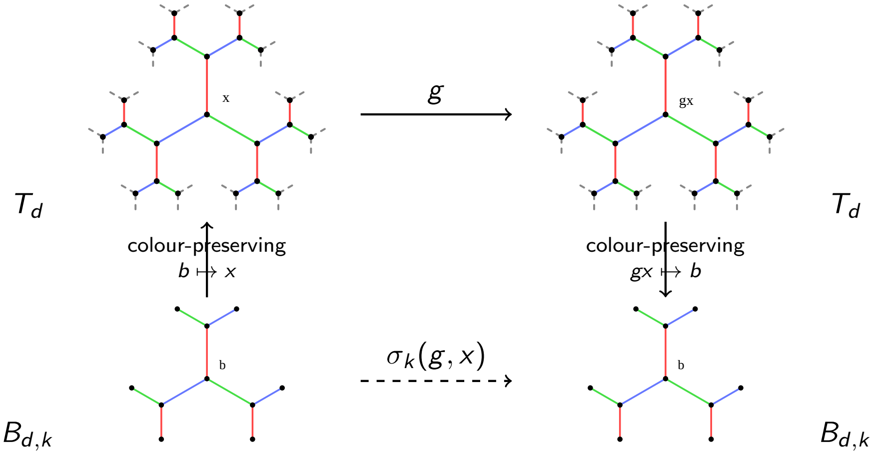


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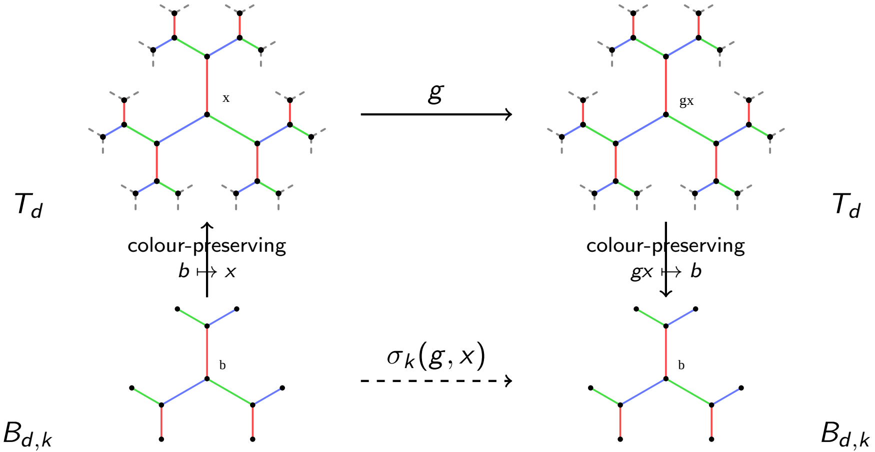


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For $F \leq \mathrm{Aut}(B_{d,k})$, set $\mathrm{U}_k(F) := \{g \in \mathrm{Aut}(T_d) \mid \forall x \in V(T_d) : \sigma_k(g,x) \in F\}$.

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## Question

Let $F \leq \mathrm{Aut}(B_{d,k})$. For $x \in V(T_d)$, what is the action that $\mathrm{U}_k(F)_x$ induces on $B(x,k)$?

# The compatibility condition (C)

## Definition

Let $F \leq \mathrm{Aut}(B_{d,k})$. Then $\mathrm{U}_k(F) \leq \mathrm{Aut}(T_d)$ satisfies (C) if and only if for all $x \in V(T_d)$ the actions $\mathrm{U}_k(F)_x \curvearrowright B(x,k)$ and $F \curvearrowright B_{d,k}$ are isomorphic.

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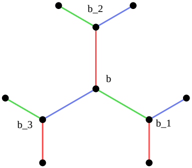

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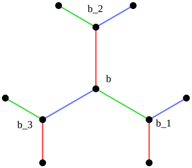


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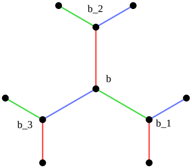


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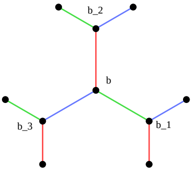


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$$\exists\ (\alpha_i,(\ ?,\ldots,\ ?\ ,\alpha,\ ?\ ,\ldots,\ ?\ )) \in F.$$

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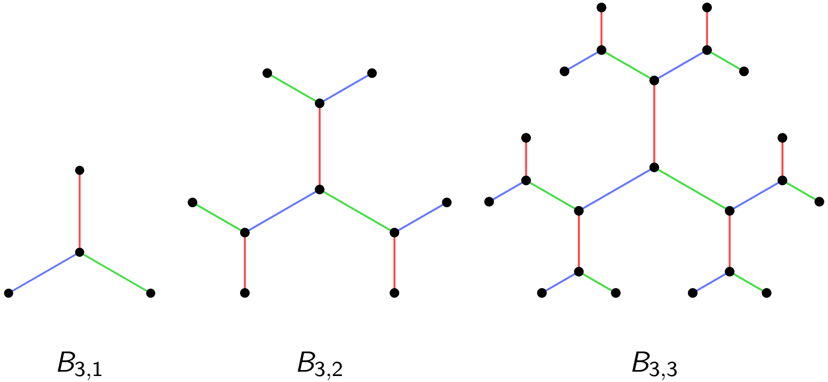

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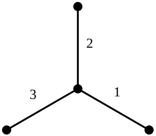
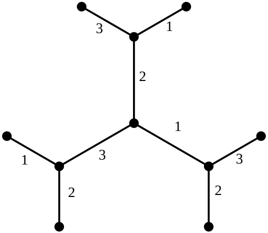
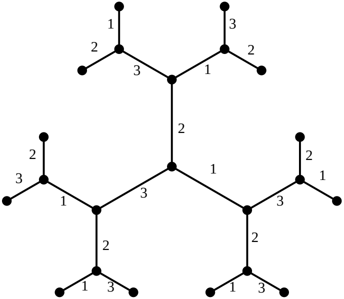



$B_{3,1}$




$B_{3,2}$

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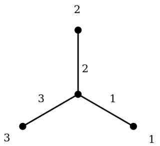


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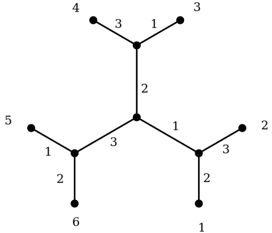


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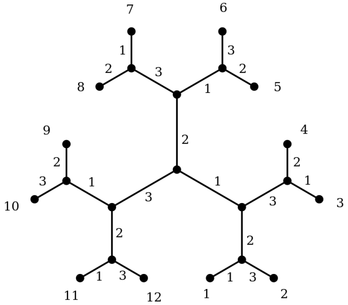


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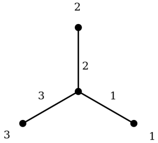


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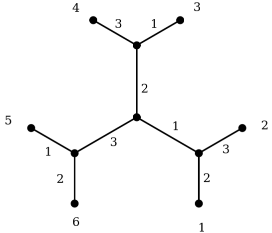


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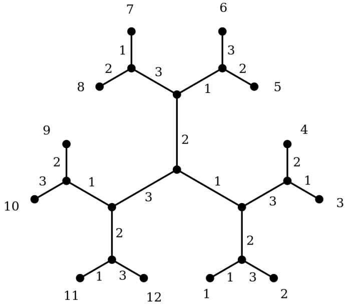


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github.com/torniers/UGALY