Discovering Symmetry

Stephan Tornier

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

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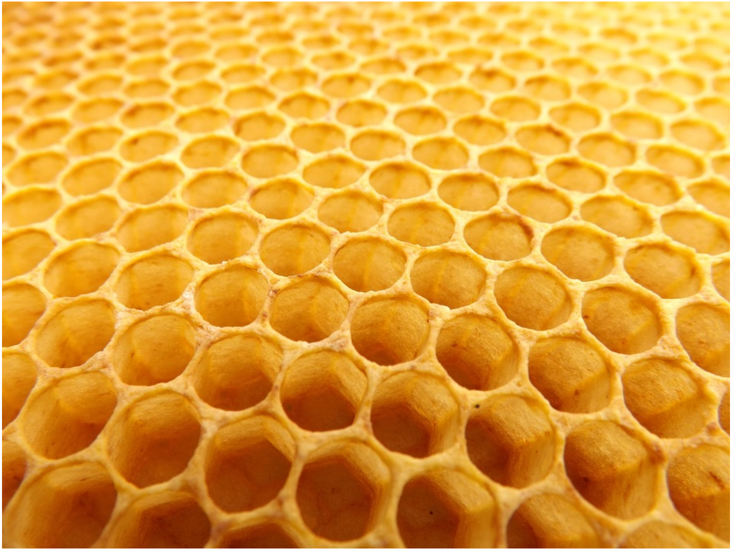
Symmetry

Beauty

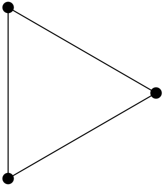



Efficiency

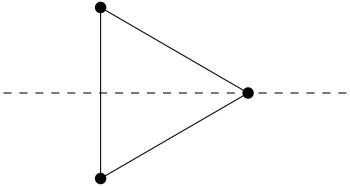


Cost-effectiveness

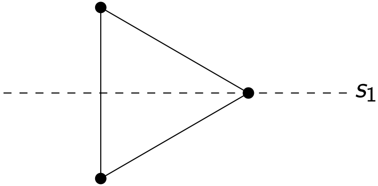
Dynamical approach



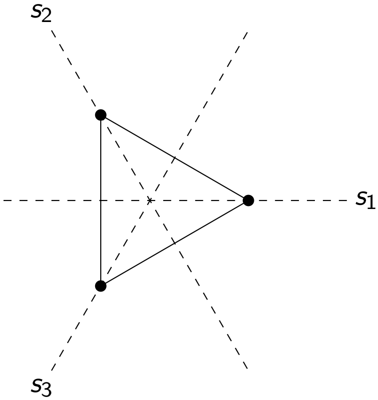
Dynamical approach



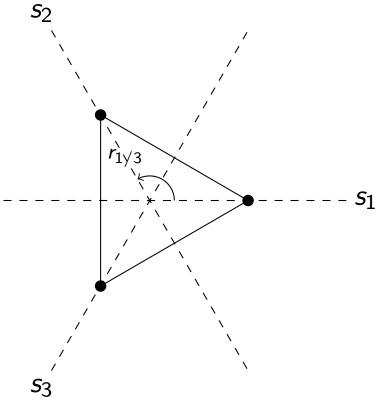
Dynamical approach



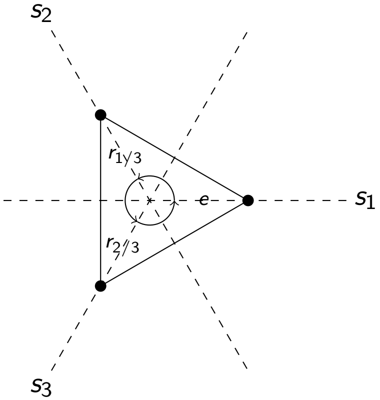
Dynamical approach



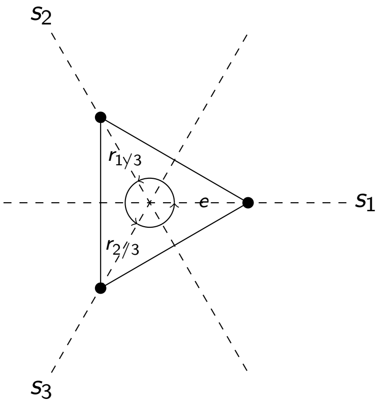
Dynamical approach



Dynamical approach

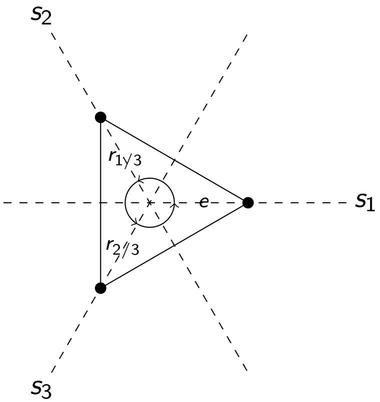


Dynamical approach



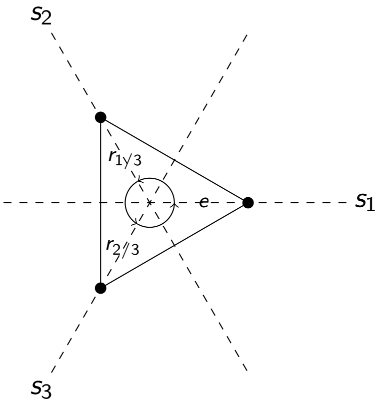
	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
e						
s_1						
s_2						
s_3						
$r_{1/3}$						
$r_{2/3}$						

Dynamical approach



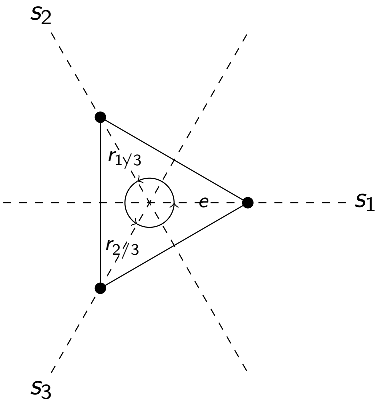
	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
e	e					
s_1						
s_2						
s_3						
$r_{1/3}$						
$r_{2/3}$						

Dynamical approach



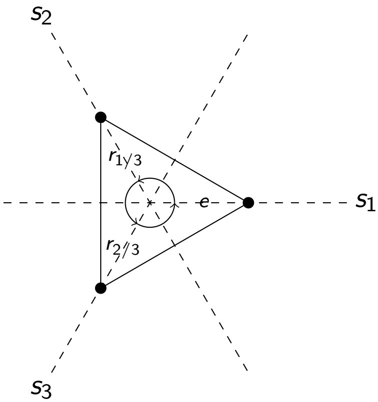
	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
e	e	s_1				
s_1						
s_2						
s_3						
$r_{1/3}$						
$r_{2/3}$						

Dynamical approach



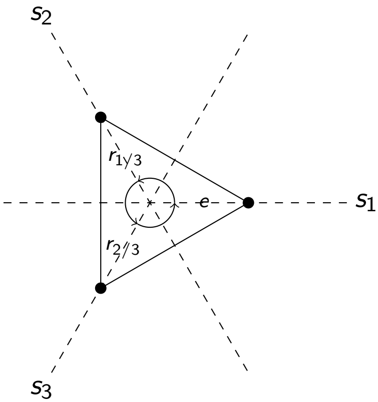
	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
e	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
s_1						
s_2						
s_3						
$r_{1/3}$						
$r_{2/3}$						

Dynamical approach



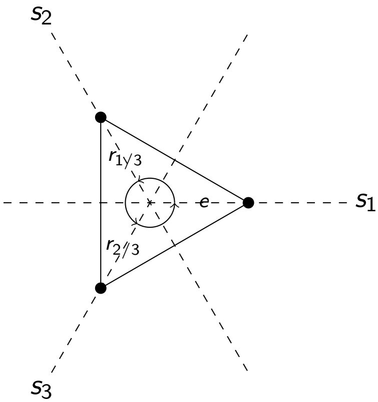
	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
e	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
s_1	s_1					
s_2	s_2					
s_3	s_3					
$r_{1/3}$	$r_{1/3}$					
$r_{2/3}$	$r_{2/3}$					

Dynamical approach



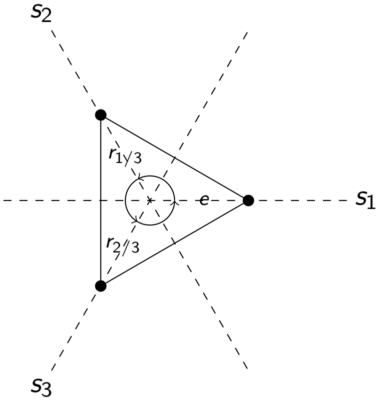
	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
e	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
s_1	s_1	e				
s_2	s_2					
s_3	s_3					
$r_{1/3}$	$r_{1/3}$					
$r_{2/3}$	$r_{2/3}$					

Dynamical approach



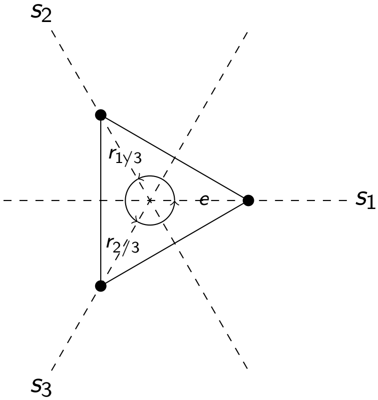
	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
e	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
s_1	s_1	e				
s_2	s_2		e			
s_3	s_3			e		
$r_{1/3}$	$r_{1/3}$					
$r_{2/3}$	$r_{2/3}$					

Dynamical approach



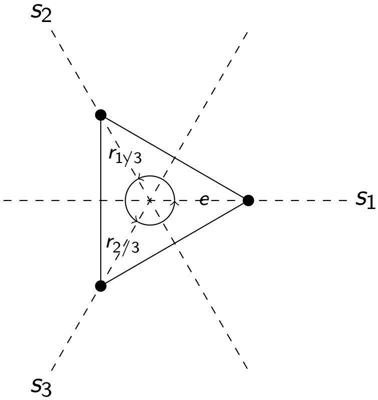
	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
e	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
s_1	s_1	e				
s_2	s_2		e			
s_3	s_3			e		
$r_{1/3}$	$r_{1/3}$				$r_{2/3}$	
$r_{2/3}$	$r_{2/3}$					

Dynamical approach



	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
e	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
s_1	s_1	e				
s_2	s_2		e			
s_3	s_3			e		
$r_{1/3}$	$r_{1/3}$				$r_{2/3}$	e
$r_{2/3}$	$r_{2/3}$				e	

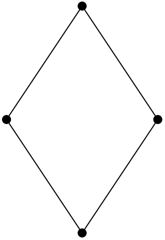
Dynamical approach

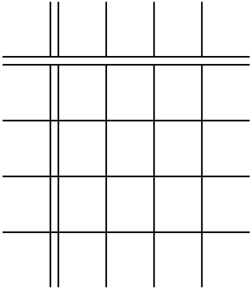


	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
e	e	s_1	s_2	s_3	$r_{1/3}$	$r_{2/3}$
s_1	s_1	e				
s_2	s_2		e			
s_3	s_3			e		
$r_{1/3}$	$r_{1/3}$				$r_{2/3}$	e
$r_{2/3}$	$r_{2/3}$				e	$r_{1/3}$

Exercise 1

Consider the figure below. Find and give names to all its symmetries, and record their compositions in the table.





Exercise 2

Complete the following table of addition of integers modulo 4.

$+_4$	0	1	2	3
0				
1				
2				
3		0		

Exercise 3

Consider the mathematical expression below.

$$a + b + c \times d$$

For every choice of a, b, c and d, it assumes a value. For example:

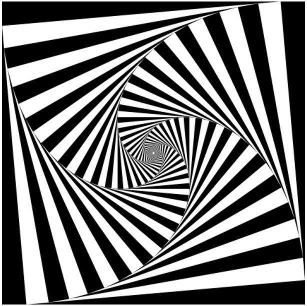
$$(-1,3,2,4) \mapsto -1+3+2\times 4 = 10,$$
$$(-1,4,2,3) \mapsto -1+4+2\times 3 = 9.$$

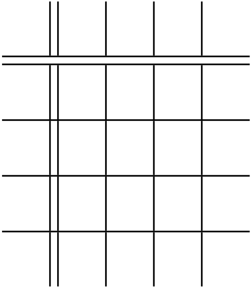
Find and give names to all rearrangements of the variables a, b, c and d that leave the value of the expression unchanged for *every* choice, and record their compositions in the table.

Note: By the above, swapping b and d is no such rearrangement.

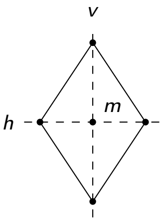
Exercise 4

Consider the image below. Find and give names to all its symmetries, and record their compositions in the table.

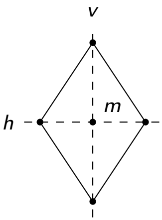




Results & comparison

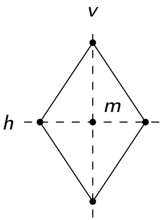


Results & comparison



	e	h	v	m
e	e	h	v	m
h	h	e	m	v
v	v	m	e	h
m	m	v	h	e

Results & comparison

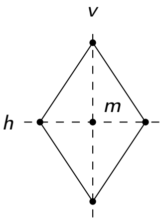


	e	h	v	m
e	e	h	v	m
h	h	e	m	v
v	v	m	e	h
m	m	v	h	e

Addition

modulo 4

Results & comparison



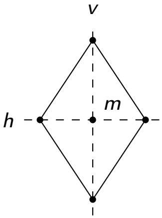
	e	h	v	m
e	e	h	v	m
h	h	e	m	v
v	v	m	e	h
m	m	v	h	e

Addition

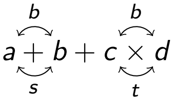
modulo 4

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Results & comparison



	e	*h*	*v*	*m*
e	*e*	*h*	*v*	*m*
h	*h*	*e*	*m*	*v*
v	*v*	*m*	*e*	*h*
m	*m*	*v*	*h*	*e*

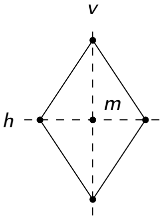


Addition

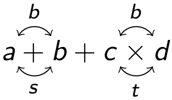
modulo 4

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Results & comparison



	e	h	v	m
e	e	h	v	m
h	h	e	m	v
v	v	m	e	h
m	m	v	h	e



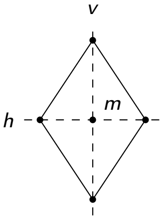
	e	s	t	b
e	e	s	t	b
s	s	e	b	t
t	t	b	e	s
b	b	t	s	e

Addition

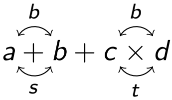
modulo 4

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Results & comparison



	e	h	v	m
e	e	h	v	m
h	h	e	m	v
v	v	m	e	h
m	m	v	h	e

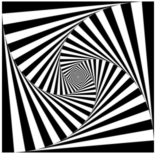


	e	s	t	b
e	e	s	t	b
s	s	e	b	t
t	t	b	e	s
b	b	t	s	e

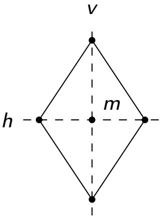
Addition

modulo 4

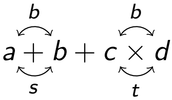
	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2



Results & comparison



	e	h	v	m
e	e	h	v	m
h	h	e	m	v
v	v	m	e	h
m	m	v	h	e

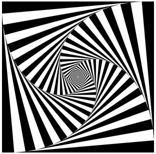


	e	s	t	b
e	e	s	t	b
s	s	e	b	t
t	t	b	e	s
b	b	t	s	e

Addition

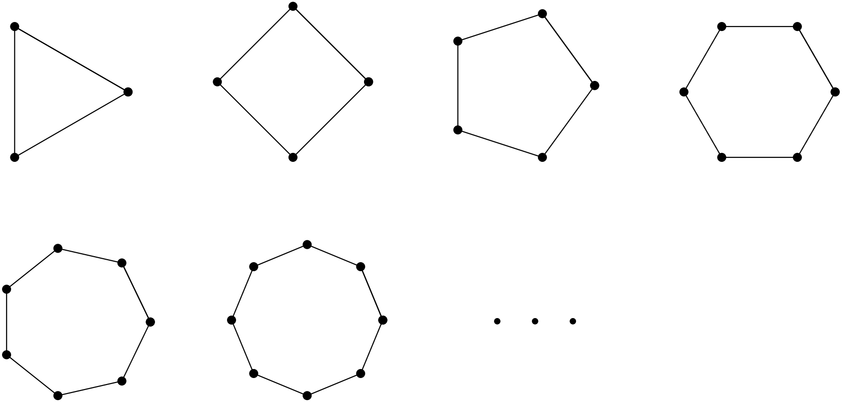
modulo 4

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

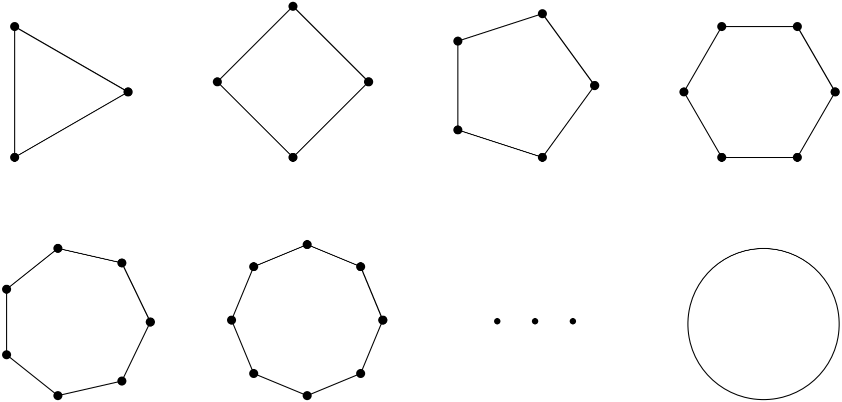


	e	$r_{1/4}$	$r_{2/4}$	$r_{3/4}$
e	e	$r_{1/4}$	$r_{2/4}$	$r_{3/4}$
$r_{1/4}$	$r_{1/4}$	$r_{2/4}$	$r_{3/4}$	e
$r_{2/4}$	$r_{2/4}$	$r_{3/4}$	e	$r_{1/4}$
$r_{3/4}$	$r_{3/4}$	e	$r_{1/4}$	$r_{2/4}$

More symmetry



More symmetry

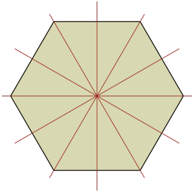


Symmetry in Mathematics

Group Theory

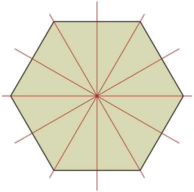
Symmetry in Mathematics

Group Theory



Symmetry in Mathematics

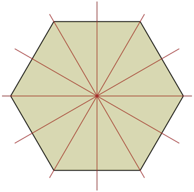
Group Theory



D_6

Symmetry in Mathematics

Group Theory

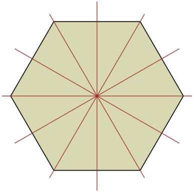


D_6

Linear Algebra

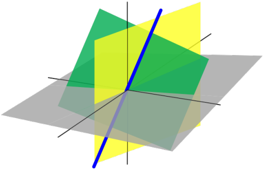
Symmetry in Mathematics

Group Theory



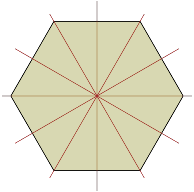
D_6

Linear Algebra



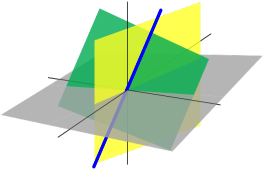
Symmetry in Mathematics

Group Theory



D_6

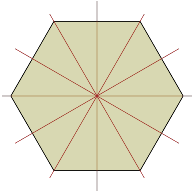
Linear Algebra



$\mathrm{GL}(\mathrm{n}, \mathbb{R})$

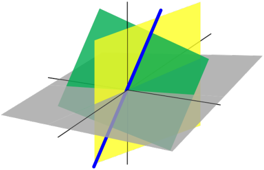
Symmetry in Mathematics

Group Theory



D_6

Linear Algebra

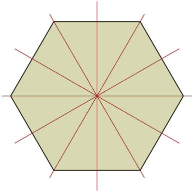


$\mathrm{GL}(\mathrm{n}, \mathbb{R})$

Number Theory

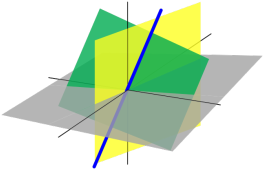
Symmetry in Mathematics

Group Theory



D_6

Linear Algebra



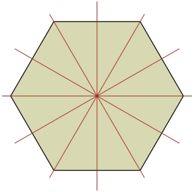
$\mathrm{GL}(n, \mathbb{R})$

Number Theory

$K \subseteq E$ fields

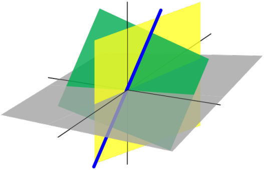
Symmetry in Mathematics

Group Theory



D_6

Linear Algebra



$\mathrm{GL}(\mathrm{n}, \mathbb{R})$

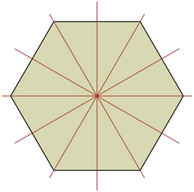
Number Theory

$K \subseteq E$ fields

$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$

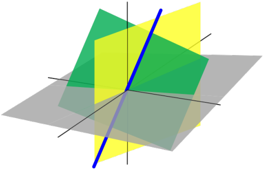
Symmetry in Mathematics

Group Theory



D_6

Linear Algebra



$\mathrm{GL}(\mathrm{n}, \mathbb{R})$

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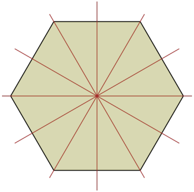

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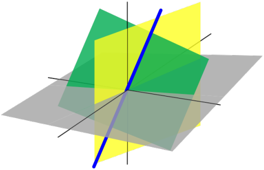

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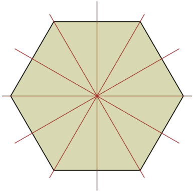

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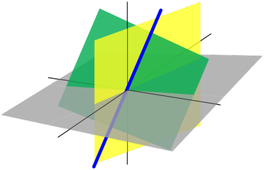

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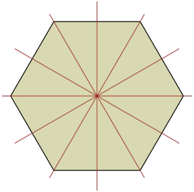

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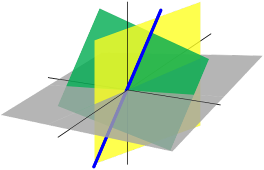

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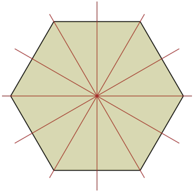

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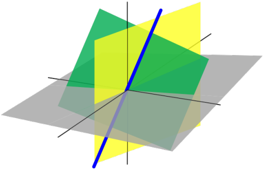

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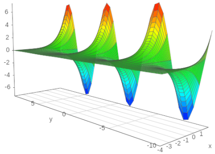

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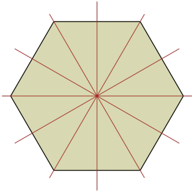

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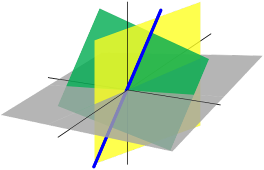

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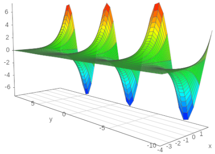

$\mathrm{O}(n)$

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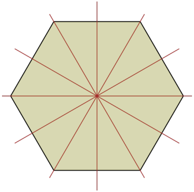

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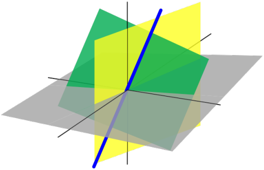

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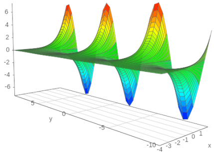

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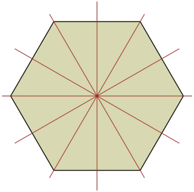

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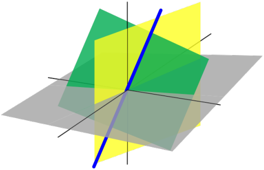

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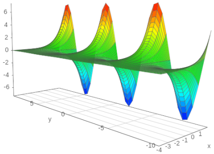

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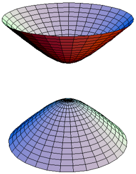

Symmetry in Mathematics

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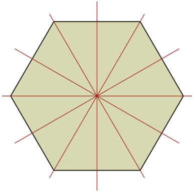

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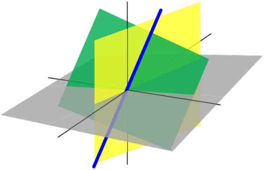

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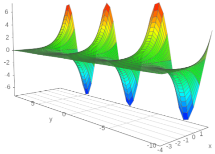

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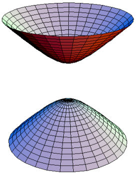

$\mathrm{O}(1, n)$

Symmetry in Mathematics

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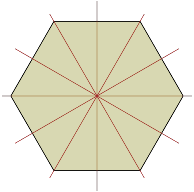

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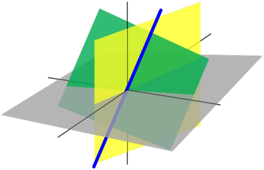

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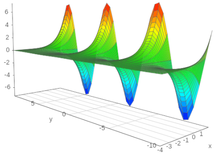

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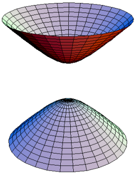

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Graph Theory

Symmetry in Mathematics

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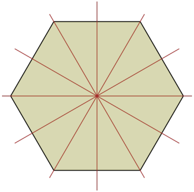

D_6

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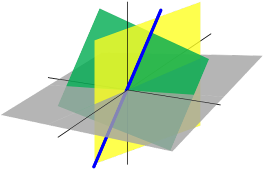

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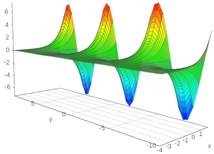

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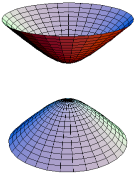

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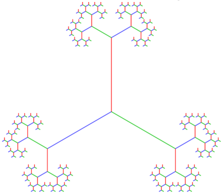

Symmetry in Mathematics

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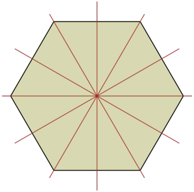

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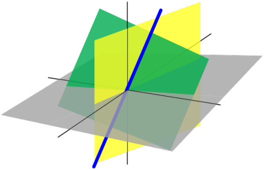

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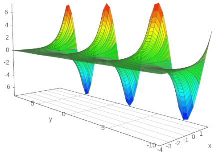

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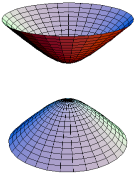

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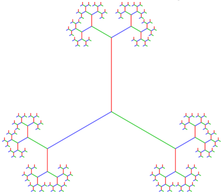

$\mathrm{Aut}(T_d)$

Symmetry in Biology

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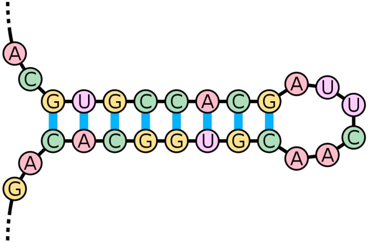
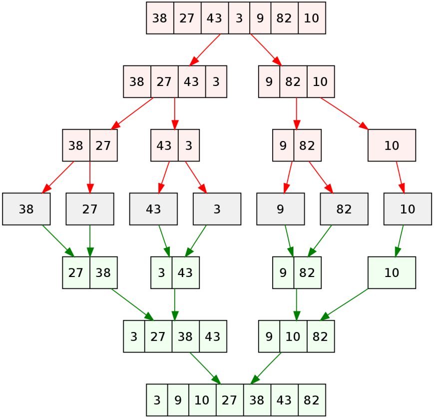


Symmetry in Computer Science



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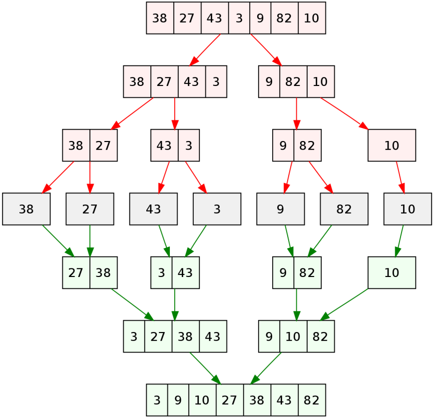
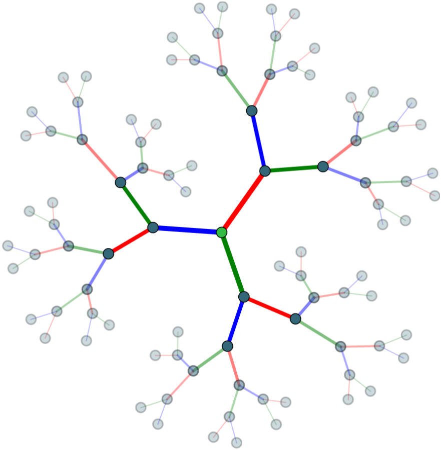
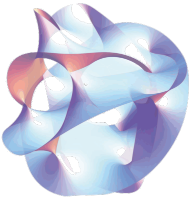
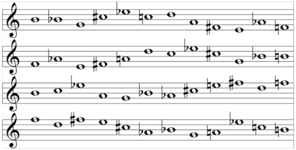




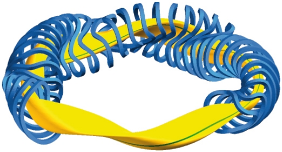
And everywhere else...



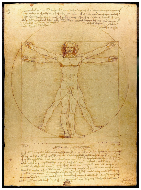
Physics



Music



Engineering



Evolution

Information Technology

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