A GAP package for self-replicating groups

Stephan Tornier

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

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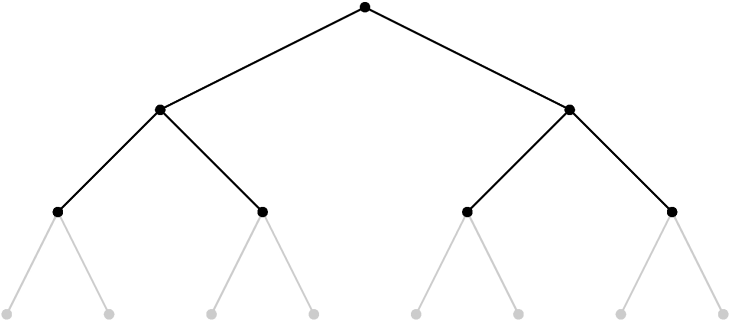
Overview

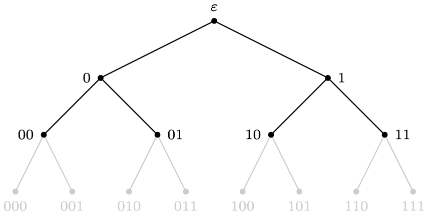


Related concepts: self-replicating, self-similar, Property R, Property R_n, finitely constrained, branch, ...

Related researchers: Bartholdi, Grigorchuk, Gupta, Horadam, Nekrashevych, Sidki, Sunik, Willis, ...

Definition

Let X be a set of size $k \in \mathbb{N}_{\geq 2}$ and let $T_k^{(r)}$ denote the k-regular rooted tree. Label the vertices of $T_k^{(r)}$ by the words X^* over the alphabet X. Fix $x_0 \in X$.



Definition

A subgroup $G \leq \mathrm{Aut}(T_k^{(r)})$ is *self-replicating* if

- G acts transitively on $X \subset X^* = V(T_k^{(r)})$
- the map $\psi : G_{x_0} \to \mathrm{Aut}(T_k^{(r)})$, $g \mapsto (w \mapsto g(x_0 w))$ has image G

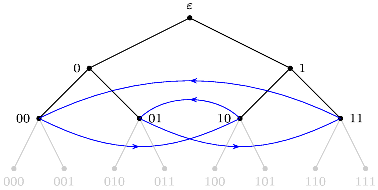
Examples

Let p be a prime and $X = \{0, \ldots, p-1\}$.

There is a bijection $\partial T_p^{(r)} \cong \mathbb{Z}_p$:

$$\omega = (w_1, w_2, \ldots) \mapsto \sum_{i=0}^{\infty} w_{i+1} p^i$$

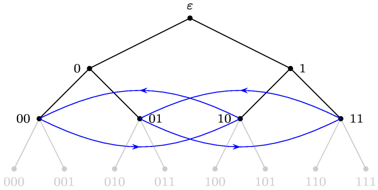
The action $\pi : \mathbb{Z}_p \curvearrowright \partial T_p^{(r)} \cong \mathbb{Z}_p$ given by $\pi(a)\omega := a + \omega$ yields a self-replicating subgroup $\mathbb{Z}_p \leq \mathrm{Aut}(T_p^{(r)})$.
The stabilizer of 0 in $\mathbb{Z}_p$ is $p\mathbb{Z}_p$.



There is a bijection $\partial T_p^{(r)} \cong \mathbb{F}_p[[t]]$:

$$\omega = (w_1, w_2, \ldots) \mapsto \sum_{i=0}^{\infty} w_{i+1} t^i$$

Then $\pi : \mathbb{F}_p[[t]] \curvearrowright \partial T_p^{(r)} \cong \mathbb{F}_p[[t]]$ given by $\pi(a)\omega := a + \omega$ yields a self-replicating subgroup of $\mathrm{Aut}(T_p^{(r)})$.
The stabilizer of 0 in $\mathbb{F}_p[[t]]$ is $t\mathbb{F}_p[[t]]$.



Greater Significance

Theorem (Baumgartner-Willis '04, Willis '20)

G *t.d.l.c.*, $\alpha \in \mathrm{Aut}(G)$, $s(\alpha^{-1}) > 1$ $\underset{G=P,\ \alpha:\ \text{translation}}{\overset{\text{tree representation}}{\rightleftharpoons}}$ $P \leq \mathrm{Aut}(T_{s(\alpha^{-1})+1})_\omega$ *vertex-transitive* $\underset{\varphi_{gv} \circ g \circ \varphi_v^{-1} \in \widehat{P}}{\overset{P_v|_{T_v}}{\rightleftharpoons}}$ $\widehat{P} \leq \mathrm{Aut}(T^{(r)}_{s(\alpha^{-1})})$ *self-replicating*

In the case of $\mathbb{Z}_p \leq \mathrm{Aut}(T_p^{(r)})$:

- $G = (\mathbb{Q}_p, +)$
- $\alpha : x \mapsto px$
- $s(\alpha^{-1}) = p$
- $P = \mathbb{Q}_p \rtimes \langle \alpha \rangle$
- $\widehat{P} = \mathbb{Z}_p$

In the case of $\mathbb{F}_p[[t]] \leq \mathrm{Aut}(T_p^{(r)})$:

- $G = (\mathbb{F}_p((t)), +)$
- $\alpha : x \mapsto tx$
- $s(\alpha^{-1}) = p$
- $P = \mathbb{F}_p((t)) \rtimes \langle \alpha \rangle$
- $\widehat{P} = \mathbb{F}_p[[t]]$

Also: intermediate growth, non-elementary amenable groups, applications to just infinite groups, fractals, ...

Reduction to finite trees

For $n \in \mathbb{N}$, let $T_{k,n}^{(r)}$ denote the k-regular rooted tree of depth n.
Label the vertices of $T_{k,n}^{(r)}$ by the words $X_n^* = \{w \in X^* \mid l(w) \leq n\}$. Fix $x_0 \in X$.

Definition

A subgroup $G \leq \mathrm{Aut}(T_{k,n}^{(r)})$ is *self-replicating* if

- G acts transitively on $X \subset X_n^* = V(T_{k,n}^{(r)})$
- the map $\psi : G_{x_0} \to \mathrm{Aut}(T_{k,n-1}^{(r)})$, $g \mapsto (w \mapsto g(x_0 w))$ has image $\mathrm{res}_{n-1}(G)$

Theorem (Horadam '15)

If $G \leq \mathrm{Aut}(T_k^{(r)})$ is self-replicating then so is $\mathrm{res}_n(G) \leq \mathrm{Aut}(T_{k,n}^{(r)})$ for all $n \in \mathbb{N}$.
Suitable inverse limits of self-replicating groups $G_n \leq \mathrm{Aut}(T_{k,n}^{(r)})$ are self-replicating.

The GAP package

(initiated by G. Willis, joint work with S. King and S. Shotter)

Current objectives are:

- provide basic methods for $G \leq \mathrm{Aut}(T_{k,n}^{(r)})$
- establish a library of self-replicating subgroups of $\mathrm{Aut}(T_{k,n}^{(r)})$
- provide methods to search the library
- identify known families and constructions in the library
- provide methods to visualise the library

Any other suggestions or comments?