Groups acting on trees from finite permutation groups 07/07/22

(Graz, 45 minutes)

Why groups acting on trees?

$$
\begin{tikzcd}
 & & \pi_1(\Gamma) \text{ discrete} \arrow[d, tail] & \\
 & & \widetilde{G/G_0} \arrow[r] \arrow[d] & \mathrm{Aut}(T_d) \arrow[d, dotted] & T_d \text{ d-regular tree} \\
G_0 \arrow[r, tail] & G \arrow[r, two heads] & G/G_0 \arrow[r] & \mathrm{Aut}(\Gamma) & \text{(possibly compact kernel)}
\end{tikzcd}
$$

G: locally compact, compactly generated

Γ regular connected graph (Cayley-Abels)

How? Local-to-global arguments (after Burger-Mozes, '00)

Let $H \leq \mathrm{Aut}(T_d)$ and $x \in V(T_d)$. Then H_x induces a permutation group on $E(x) = \{ e \in E(T_d) \mid o(e) = x \}$. We say that H is locally "X" if said permutation group has "X" for every vertex $x \in V(T_d)$.

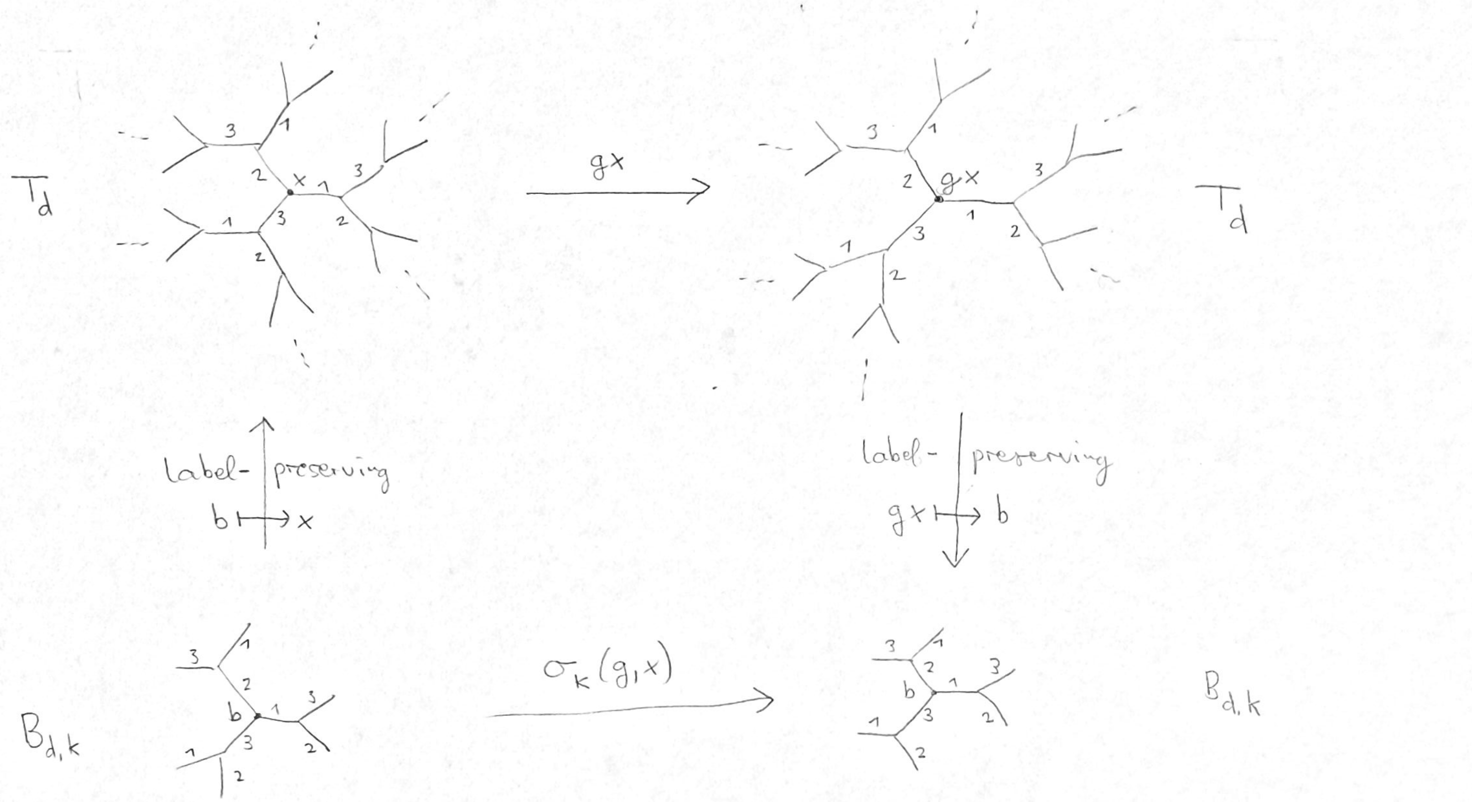
Def. Let $H \leq \mathrm{Aut}(T_d)$. Define $QZ(H) := \{ h \in H \mid C_H(h) \text{ is open} \}$.

Thm. (T. '18) Let $H \leq \mathrm{Aut}(T_d)$ be non-discrete. If H is locally

(i) transitive then $QZ(H)$ contains no edge-inversion.
(ii) semiprimitive then $QZ(H)$ contains no edge-fixating element.
(iii) quasiprimitive then $QZ(H)$ contains no vertex-fixating element.
(iv) k-transitive ($k \in \mathbb{N}$) then $QZ(H)$ contains no translation of length k.

This theorem is essentially sharp.
Counterexamples are constructed using the following generalisation of Burger-Mozes universal groups.

Universal groups



Def. Let $F \leq \mathrm{Aut}(B_{d,k})$. Define $U_k(F) := \{ g \in \mathrm{Aut}(T_d) \mid \forall x : \sigma_k(g,x) \in F \}$

Ex. · $U_k(\mathrm{Aut}(B_{d,k})) = \mathrm{Aut}(T_d)$

· $U_k(\{\mathrm{id}\}) = \{ g \in \mathrm{Aut}(T_d) \mid g$ is label-preserving $\}$

$\cong \mathbb{Z}/2\mathbb{Z} * \cdots * \mathbb{Z}/2\mathbb{Z}$ (d factors)

(Label-preserving edge inversions, Cayley graph

Q. Given $x \in V(T_d)$, how does $U_k(F)_x$ act on the ball $B(x,k)$?

We say that $F \leq \mathrm{Aut}(B_{d,k})$ satisfies the compatibility condition (C) if this action is precisely F (rather than a strict subgroup of F).

$$\mathrm{Aut}(B_{d,k}) \lesssim \mathrm{Aut}(B_{d,k-1}) \ltimes \prod_{i=1}^{d} \mathrm{Aut}(B_{d,k-1})$$

$$g \longmapsto (\sigma_{k-1}(g,b), (\sigma_{k-1}(g,b_1), \ldots, \sigma_{k-1}(g,b_d)))$$

(C) $\forall i \in \{1,\ldots,d\}\ \forall (\alpha, (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_d)) \in F$

$\exists (\alpha_i, (?, \ldots, ?, \alpha, ?, \ldots, ?)) \in F$

This can be handled computationally $\rightarrow$ UGALY.

Ex. $(k=2)$ Let $F \leq \mathrm{Sym}(\{1,\dots,d\})$

- $\Gamma(F) := \{(a,(a,\dots,a)) \mid a \in F\} \cong F$

- $\Phi(F) := \{(a,(a_1,\dots,a_d)) \mid a, a_i \in F,\ a_i(i) = a(i)\} \cong F \ltimes \prod_{i=1}^{d} F_i$

 (this one is maximal with (C) and projecting to F)

- Suppose F preserves a partition $P: \{1,\dots,d\} = \bigsqcup_{i \in I} P_i$.

 $\Phi(F,P) := \{(a,(a_1,\dots,a_d)) \in \Phi(F) \mid k,l \in P_i \Rightarrow a_k = a_l\} \cong F \ltimes \prod_{i \in I} F_{P_i}$

- ... normal subgroups of stabilisers of F, abelian quotients of F, ...

One more concept (Banks-Elder-Willis '15)

Let $H \leq \mathrm{Aut}(T_d)$. Define the P_k-closure of H:

$$H^{(P_k)} := \{g \in \mathrm{Aut}(T_d) \mid \forall x \in V(T_d)\ \exists h \in H : g|_{B(x,k)} = h|_{B(x,k)}\}$$

Then: $H^{(P_1)} \geq H^{(P_2)} \geq \dots \geq H^{(P_k)} \geq \dots \geq \overline{H} \geq H$, and $\bigcap_{k \in \mathbb{N}} H^{(P_k)} = \overline{H}$.

If $H^{(P_k)} = H$, we say that H is P_k-closed.

For example, $U_k(F)$ is P_k-closed.

Thm. (T. '20)

$$\left\{ H \leq \mathrm{Aut}(T_d) \;\middle|\; \begin{array}{c} \text{locally transitive} \\ \text{inversion of order 2} \\ P_k\text{-closed} \end{array} \right\} \xleftrightarrow[U_k(F)]{1:1} \left\{ F \leq \mathrm{Aut}(B_{d,k}) \;\middle|\; \begin{array}{c} \text{locally transitive} \\ \text{condition (C)} \end{array} \right\}$$

Note: strategy to classify all locally transitive groups with an inversion of order 2.

Idea of proof:

Use local transitivity and the order 2 inversion to construct a labelling of T_d such that $H \geq U_1(\{id\}) \cong \mathbb{Z}/2\mathbb{Z} * \dots * \mathbb{Z}/2\mathbb{Z}$. Then let F be the action of vertex-stabilisers on k-balls.