Compatibility Cocycles of Finite Permutation Groups

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What is the size and structure of the set of compatibility cocycles of G?
What about the subset of involutive compatibility cocycles of G?

$\lvert\Omega\rvert$	Permutation Group	$\lvert\mathrm{ICC}(G)\rvert$	$\lvert\mathrm{CC}(G)\rvert$

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2	S_2	1	1
3	$C_3 = A_3$	1	1
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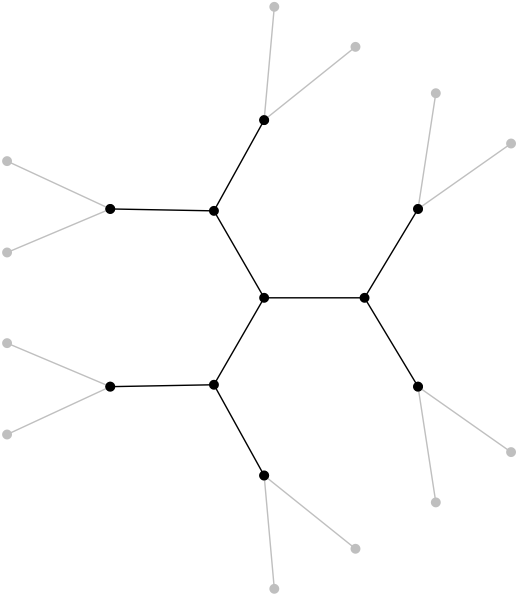
4	C_4	1	1
4	$C_2 \times C_2$	1	1
4	D_4	8	16
4	A_4	28	81
4	S_4	256	2160
5	C_5	1	1
5	D_5	16	32
5	$\mathrm{AGL}(1,5)$	272	1024
5	A_5	?	?
5	S_5	?	?
6	D_6	32	64
7	D_7	64	128

Motivation



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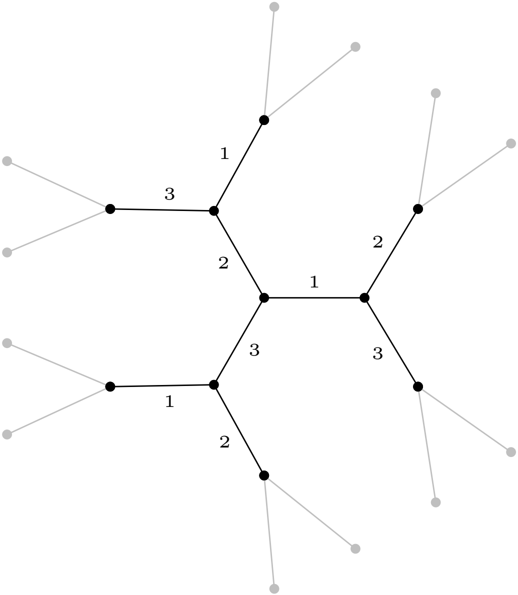

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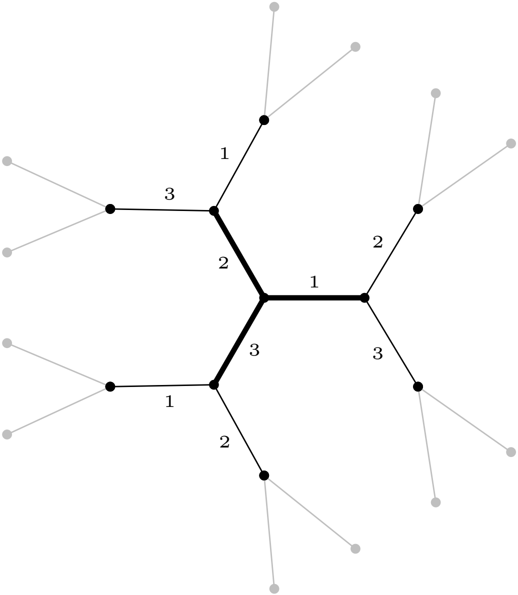

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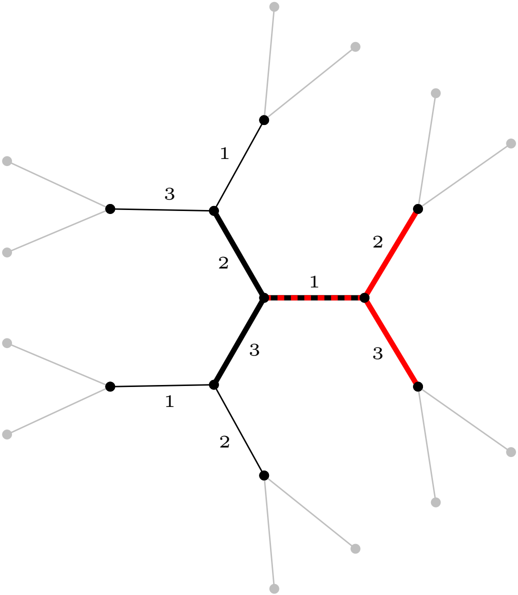

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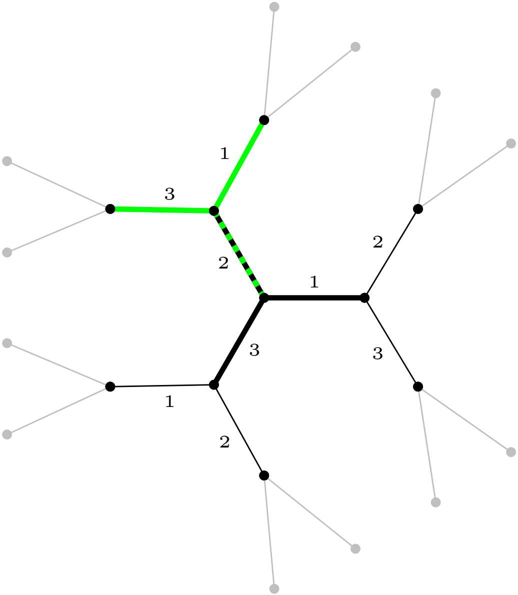

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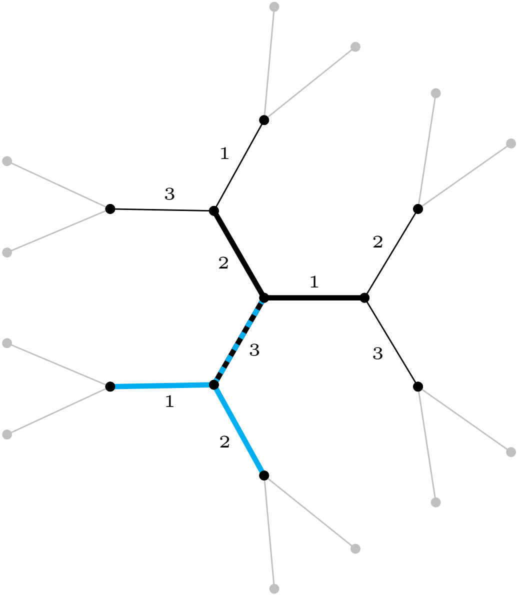

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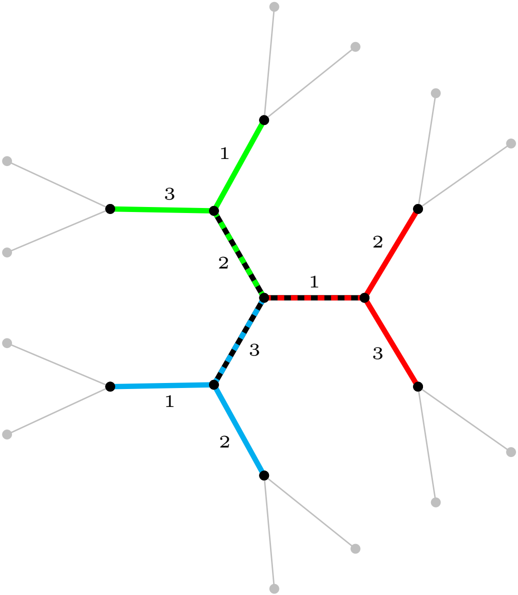

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$$z^a(g,\omega) = az(a^{-1}ga, a^{-1}\omega)a^{-1}, \text{ and}$$
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Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group.
The map $K_G \to \mathrm{CC}(G)$ given by $k \mapsto z_0^k$ is a monoid homomorphism.

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4	A_4	28	81	$27+27+27$
4	S_4	256	2160	$216+648+1296$
5	C_5	1	1	1
5	D_5	16	32	$16+16$
5	$\mathrm{AGL}(1,5)$	272	1024	$256+256+256+256$
5	A_5	?	?	?
5	S_5	?	?	?
6	D_6	32	64	$32+32$
7	D_7	64	128	$64+64$

Left-absorbing cocycles

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Proposition (and Open Problem)

Let $G \leq \mathrm{Sym}(\Omega)$ be transitive permutation group. Then

Left-absorbing cocycles

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(i) (cocycle) $\forall g, h \in G \ \forall \omega \in \Omega : \ z(gh, \omega) = z(g, h\omega)z(h, \omega)$, and

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