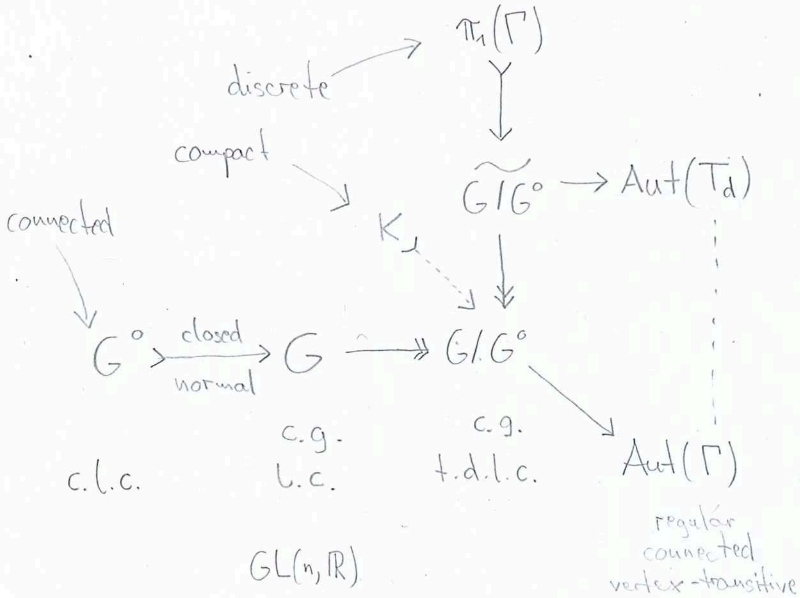
A characterisation of discrete (P)-closed groups acting on trees

(Münster, 10/10/23, 60 minutes; joint work with Marcus Chijoff)

Why groups acting on trees?



$GL(n,\mathbb{R})$

$\mathrm{Aut}(T_d)$

$PGL(n,\mathbb{Q}_p)$

$\vdots$

Why (P)-closed groups? (more generally: (P_k)-closed groups, $k \in \mathbb{N}$, $(P) = (P_1)$)

First introduced by Tits '70 to construct simple groups acting on trees
A generalisation and reformulation due to Banks–Elder–Willis '13:

Def. Let T be a tree, $H \leq \mathrm{Aut}(T)$ and $k \in \mathbb{N}$. The (P_k)-closure of H is

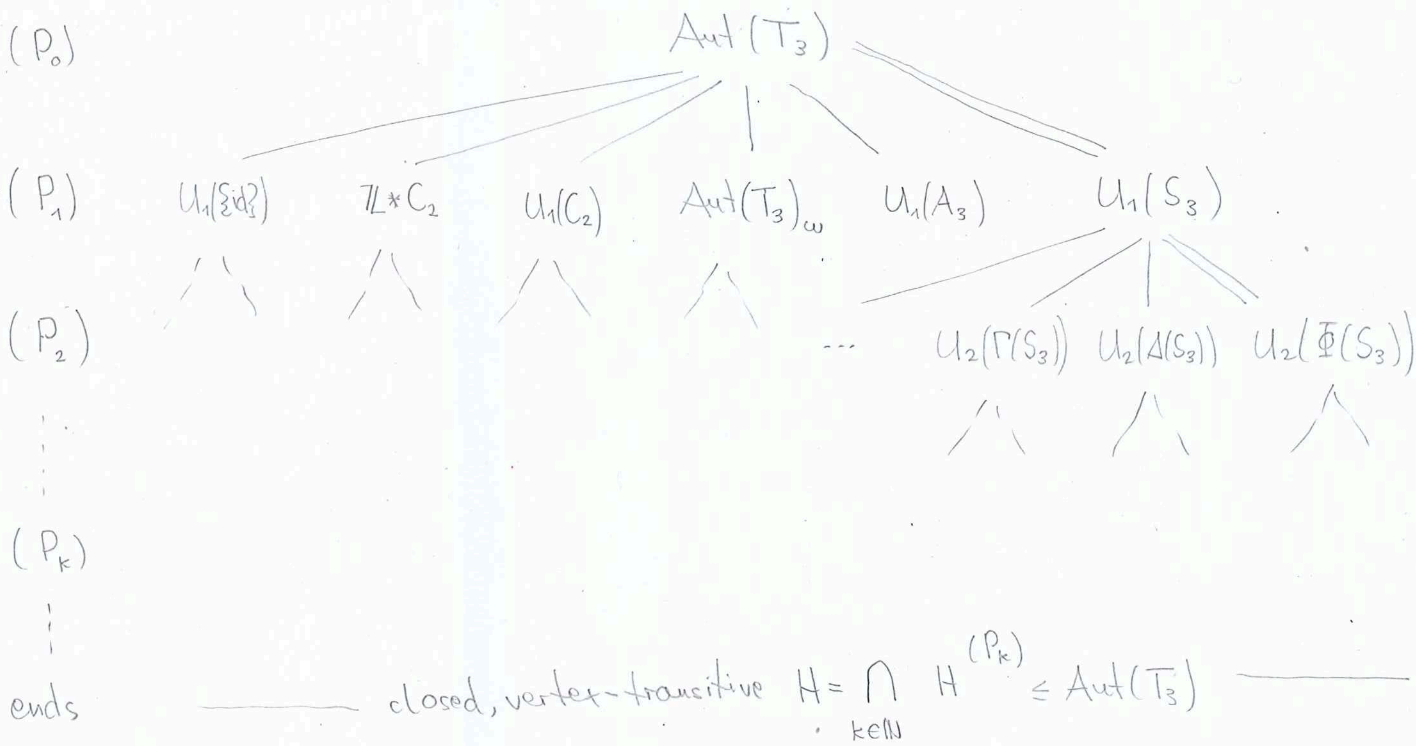
$$H^{(P_k)} := \{ g \in \mathrm{Aut}(T) \mid \forall v \in V(T) \; \exists h \in H : g|_{B(v,k)} = h|_{B(v,k)} \}.$$

The group H is (P_k)-closed, or has Property (P_k), if $H = H^{(P_k)}$.

In this situation: $H^{(P_1)} \geq H^{(P_2)} \geq \dots \geq H^{(P_k)} \geq \dots \geq H$ and $\overline{H} = \bigcap_{k \in \mathbb{N}} H^{(P_k)}$, and $H^{(P_k)}$ is (P_k)-closed.

Idea: classify all closed subgroups of $\mathrm{Aut}(T)$ by classifying all groups that can appear as $H^{(P_k)}$, i.e. exactly the (P_k)-closed groups, and form all possible intersections.

For example, in the case of $T = T_3$ and <u>vertex-transitive</u> groups



<u>Caveat</u>: need e.g. vertex-transitivity to deal with conjugacy

<u>Classification results / plans</u> (usually making some kind of transitivity assumptions)

1. <u>Local transitivity</u>

   Given $G \leq \mathrm{Aut}(T)$, the <u>local action</u> of G at $v \in V(T)$ is the permutation group $G_v \curvearrowright \{\text{arcs originating at } v\} = o^{-1}(v)$.

   - Burger–Mozes '00: locally transitive, (P_1)-closed subgroups of $\mathrm{Aut}(T_d)$ $\rightsquigarrow U(F)$
   - Smith '18: locally transitive, (P_1)-closed subgroups of $\mathrm{Aut}(T_{m,n})$ $\rightsquigarrow U(F_1, F_2)$
   - T. '18: locally transitive, (P_k)-closed subgroups of $\mathrm{Aut}(T_d)$ containing an inversion of order 2 $\rightsquigarrow U_k(F)$

2. <u>Boundary transitivity</u>

   - Radu '15: boundary 2-transitive and locally at least alternating group
   - Reid '23: ambition to weaken the A_n assumption

3. Vertex/arc-transitivity

- vertex-transitive: strategy above
- (s-)arc-transitive: lots of work, especially in the context of discrete groups

4. No transitivity assumptions

- Reid-Smith '20 $\rightsquigarrow$ (P)-closed groups (huge milestone!)
- Lehner-Lindorfer-Möller-Woess: (P_k)-closed groups, work in progress

Thm (Reid-Smith '20)

$$\left\{\begin{array}{l}\text{Pairs } (T, G)\\ G \leq \mathrm{Aut}(T)\\ (P)\text{-closed}\end{array}\right\} \Big/ \cong \quad \overset{1:1}{\longleftrightarrow} \quad \left\{\begin{array}{l}\text{Local action}\\ \text{diagrams}\end{array}\right\} \Big/ \cong$$

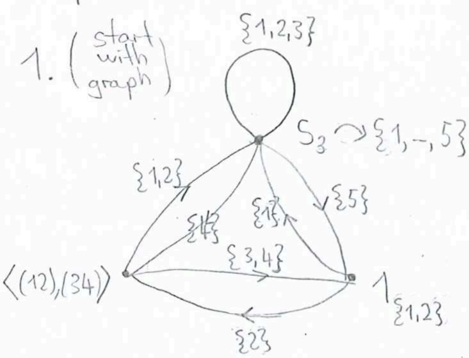
Appreciate how general this is!

Def. (local action diagram) A local action diagram is a triple $\Delta = (\Gamma, (X_a)_{a \in A(\Gamma)}, (G(v))_{v \in V(\Gamma)})$ where

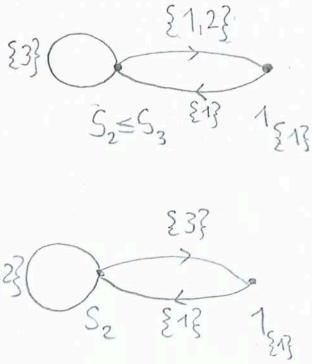
- $\Gamma = (V, A; o, t, r)$ is a connected graph
- X_a is a non-empty set
- $G(v)$ is a permutation group acting on $\bigsqcup_{o^{-1}(v)} X_a =: X_v$ whose orbits are precisely the X_a

Examples

1. (start with graph)



2. (start with groups)



3. Systematic computations

images

T_3, T_6

From a pair (T,G) to a local action diagram Δ

Let $\Gamma := T/G$ and $\pi: T \to \Gamma$ the natural projection.

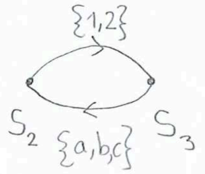
For all $v \in V(\Gamma)$, choose $\tilde{v} \in \pi^{-1}(v)$. Given $a \in A(\Gamma)$ define $(a \in o^{-1}(v))$

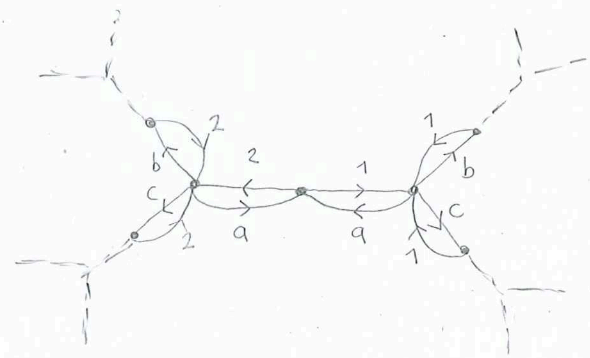
$X_a := \{ \tilde{a} \in o^{-1}(\tilde{v}) \mid \pi(\tilde{a}) = a \}$. Set $G(v)$ to be the local action of G at $\tilde{v}$.

From a local action diagram Δ to a pair (T,G)

Given Δ we construct an arc-labelled tree T and a group G acting on it.

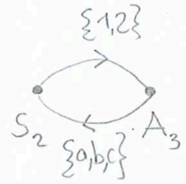
By example:





$T_{2,3}$

$U(\Delta) = U(S_2, S_3)$



same tree

$U(\Delta) = U(S_2, A_3)$

Notice that T comes with a labelling $l: A(T) \longrightarrow \bigsqcup_{a \in A(\Gamma)} X_a$,
so for every $v \in V(T)$ and $g \in \mathrm{Aut}_\pi(T) = \{ g \in \mathrm{Aut}(T) \mid \pi \circ g = \pi \}$ we get a local action $\sigma(g,v) := l \circ g \circ l|_{o^{-1}(v)}^{-1} \in \mathrm{Sym}(X_{\pi(v)})$.

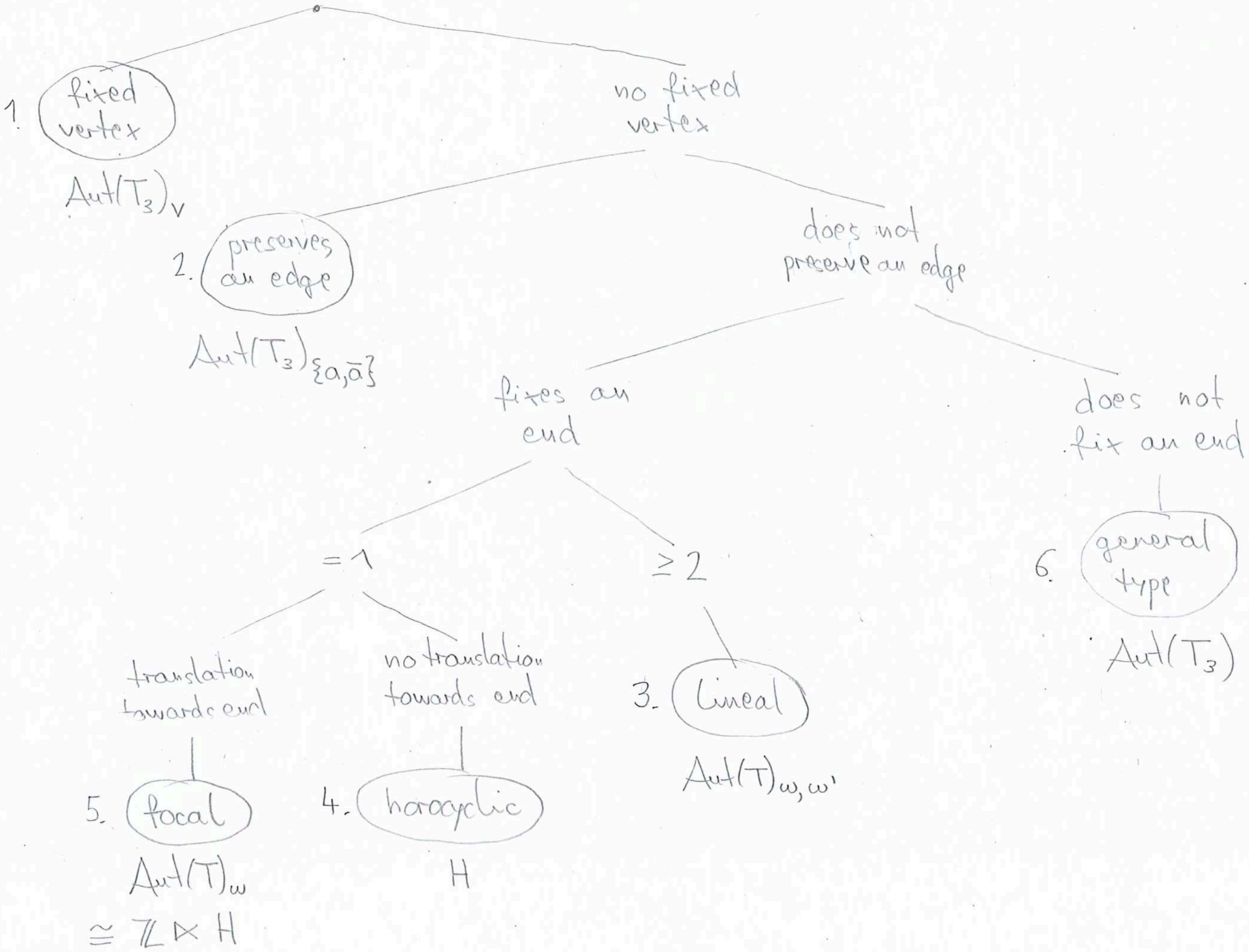
Define: $G := U(\Delta) := \{ g \in \mathrm{Aut}_\pi(T) \mid \forall v \in T: \sigma(g,v) \in G(\pi(v)) \leq \mathrm{Sym}(X_{\pi(v)}) \}$.

Powerful correspondence between properties of $U(\Delta)$ and Δ. For example:

1. {Fixed ends of $U(\Delta)$, invariant subtrees of $U(\Delta)$}
   $\overset{1:1}{\longleftrightarrow}$ {strongly confluent partial orientations of Δ}
   "scopo"
   (note: combinatorial in nature, computable when Δ is finite – GAP package)
   $U(\Delta) \curvearrowright T$ geometrically dense $\iff$ Δ has no non-trivial scopos)
2. Local compactness, compact generation of $U(\Delta)$ $\rightsquigarrow$ Tits' simplicity
   $\leftrightarrow$ condition on Δ
3. Discreteness of $U(\Delta)$ $\longleftrightarrow$ condition on Δ. $\longleftarrow$

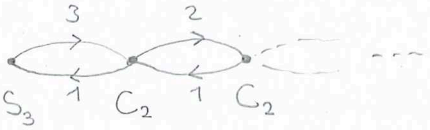
Three steps

1. Distinguish groups acting on trees into six distinct types (well-known)

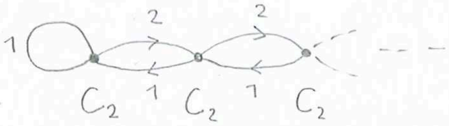


2. Determine the type of $U(\Delta)$ from Δ alone. Mostly implicit in Reid-Smith, explicit in Marcus' Honours thesis.

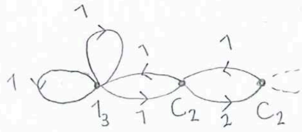
1. $\mathrm{Aut}(T_3)_v$



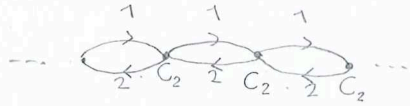
2. $\mathrm{Aut}(T_3)_{\{a,\bar{a}\}}$



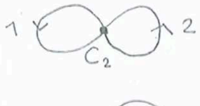
3. $\mathrm{Aut}'(T_3)_{\omega,\omega'}$



4. Horocyclic H



5. Focal $\mathrm{Aut}(T_3)_\omega$



6. General type $\mathrm{Aut}(T_3)$

conditions on Δ: shape of Γ and existence of certain scopos (computable for finite diagrams)

3. Characterising discreteness of $U(\Delta)$ in terms of Δ assuming Δ is of a given type (also characterised by Δ).

<u>Thm.</u> (Chijoff – T. '23) Let $\Delta = (\Gamma, (X_a)_a, (G(v))_v)$ be a local action diagram. Then $U(\Delta)$ is discrete if and only if exactly one of the following holds:

(i) Δ of <u>fixed vertex type</u> and there are only finitely many non-trivial $G(v)$, and each $G(v)$ has a finite base.

(ii) Δ of <u>inversion type</u> and --- same as (i)

(iii) Δ of <u>lineal type</u> and each $G(v)$ is trivial.

(iv) Δ of <u>general type</u> and $G(v)$ is semi-regular for all $v \in V(\Gamma)$ belonging to a certain subset, and $G(v)$ is trivial otherwise

In particular, there are no <u>focal</u> or <u>horocyclic</u> discrete (P)-closed groups.

(sketch of e.g. proof of (iii))

(iv) generalises result for universal groups