A characterisation of discrete (P)-closed groups acting on trees

(SODO, Auckland, 15/02/24, 60 minutes; joint work with Marcus Chijoff)

Premise: groups acting on trees are important for theoretical and practical reasons (happy to explain over morning tea)

Why (P)-closed groups? (more generally: (P_k)-closed groups, $k \in \mathbb{N}_0$)
then: $(P) = (P_1)$

First introduced by Tits ('70) to exhibit simple groups acting on trees
A generalisation and reformulation due to Banks - Elder - Willis '13:

Def. Let T be a tree, $H \leq Aut(T)$ and $k \in \mathbb{N}_0$. The (P_k)-closure of H

$$H^{(P_k)} := \{ g \in Aut(T) \mid \forall v \in V(T) \ \exists h \in H : g|_{B(v,k)} = h|_{B(v,k)} \}$$

We say that H is (P_k)-closed, or has Property (P_k) if $H = H^{(P_k)}$.

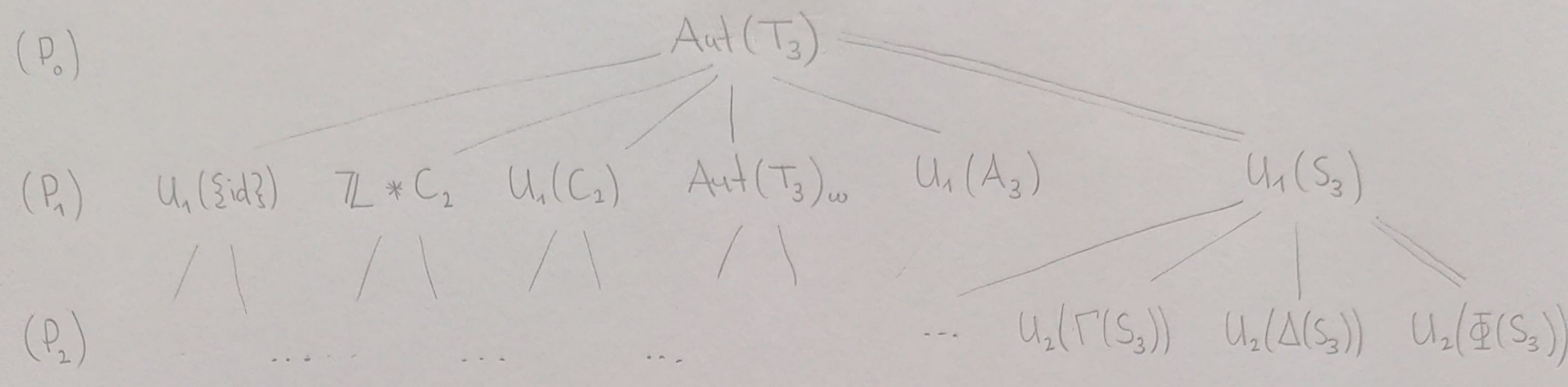
In this situation:

- $H^{(P_0)} \geq H^{(P_1)} \geq H^{(P_2)} \geq \dots \geq H^{(P_k)} \geq \dots \geq \overline{H} \geq H$
- $\bigcap_{k \in \mathbb{N}_0} H^{(P_k)} = \overline{H}$
- $(H^{(P_k)})^{(P_k)} = H^{(P_k)}$, i.e. $H^{(P_k)}$ is (P_k)-closed

Idea: classify all closed subgroups of $Aut(T)$ by classifying all groups that can appear as $H^{(P_k)}$, i.e. any (P_k)-closed group, and form all possible intersections

Caveat: to make this work up to conjugacy, we need to make an additional assumption; for example: vertex-transitivity

For example, in the case of $T = T_3$ for <u>vertex-transitive</u> groups



⋮

(P_k)

⋮

ends ———— closed, vertex-transitive $H = \bigcap_{k \in \mathbb{N}_0} H^{(P_k)} \leq \mathrm{Aut}(T_3)$ ————

<u>Classification results / plans</u> (usually making some kind of transitivity assumption)

<u>Def.</u> Let T be a tree and $G \leq \mathrm{Aut}(T)$. The local action of G at $v \in V(T)$ is the permutation group $G_v \curvearrowright \{\text{arcs originating at } v\}$.

1. <u>Local transitivity</u>
   - Burger–Mozes '00: locally transitive, (P_1)-closed subgroups of $\mathrm{Aut}(T_d)$ that contain an inversion $\rightsquigarrow U(F)$
   - Smith '18: (P_1)-closed subgroups of $\mathrm{Aut}(T_{m,n})$ preserving the bipartition $\rightsquigarrow U(F_1, F_2)$
   - T. '18: locally transitive, (P_k)-closed subgroups of $\mathrm{Aut}(T_d)$ that contain an inversion of order 2 $\rightsquigarrow U_k(F)$

2. <u>Boundary transitivity</u>
   - Rada '15: boundary 2-transitive and locally at least alternating group
   - Reid '23: towards weakening the alternating group assumption

3. Vertex / arc-transitivity

- vertex-transitive: strategy above
- (s-)arc-transitive: lots of work, especially in the context of discrete groups / Weiss conjecture

4. No transitivity assumptions

- Reid-Smith '20: (P)-closed groups (any tree) (huge milestone!)
- Lehner-Lindorfer-Möller-Woess: (P_k)-closed groups, work in progress

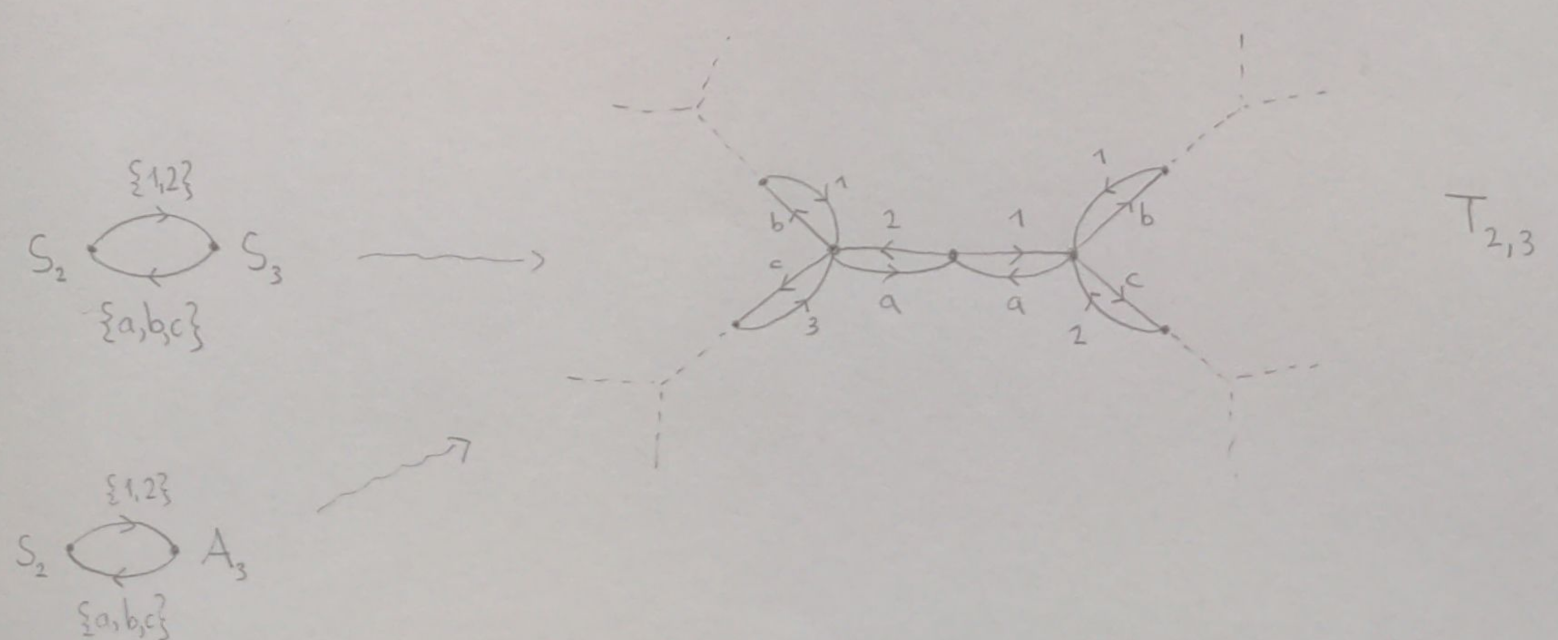
Thm. (appreciate the generality)

$$\left.\begin{array}{c}\text{Pairs } (T,G)\\ G \leq \mathrm{Aut}(T)\\ (P)\text{-closed}\end{array}\right/ \cong \quad \xleftrightarrow{1:1} \quad \left.\begin{array}{c}\text{Local action}\\ \text{diagrams}\end{array}\right/ \cong$$

Def. A local action diagram is a triple $\Delta = (\Gamma, (X_a)_{a\in A(\Gamma)}, (G(v))_{v\in V(\Gamma)})$ where

- $\Gamma = (V, A, o, t, r)$ is a connected graph
- X_a is a non-empty set
- $G(v)$ is a permutation group acting on $X_v := \bigsqcup_{a\in o^{-1}(v)} X_a$ whose orbits are precisely the X_a

From a local action diagram to a pair (T,G) (by example)



Note that T comes with a projection $\pi : T \to \Gamma$ and labelling $l : A(T) \longrightarrow \bigsqcup_{a \in A(\Gamma)} X_a$, so for $g \in \mathrm{Aut}_\pi(T) = \{ g \in \mathrm{Aut}(T) \mid \pi \circ g = \pi \}$

and $v \in V(T)$ we get a local action $\sigma(g,v) = l \circ g \circ l|_{o^{-1}(v)}^{-1} \in \mathrm{Sym}(X_{\pi(v)})$

<u>Define</u> $G := U(\Delta) := \{ g \in \mathrm{Aut}_\pi(T) \mid \forall v \in V(T) : \sigma(g,v) \in G(\pi(v)) \leq \mathrm{Sym}(X_{\pi(v)}) \}$.

<u>From a pair (T,G) to a local action diagram</u>

Let $\Gamma := G \backslash T$ and $\pi : T \to \Gamma$ the natural projection.

For all $v \in V(\Gamma)$, choose $\tilde{v} \in \pi^{-1}(v)$. Given $a \in o^{-1}(v)$ put $X_a := \{ \tilde{a} \in o^{-1}(\tilde{v}) \mid \pi(\tilde{a}) = a \}$.

Finally, let $G(v)$ be the usual local action at $\tilde{v}$.

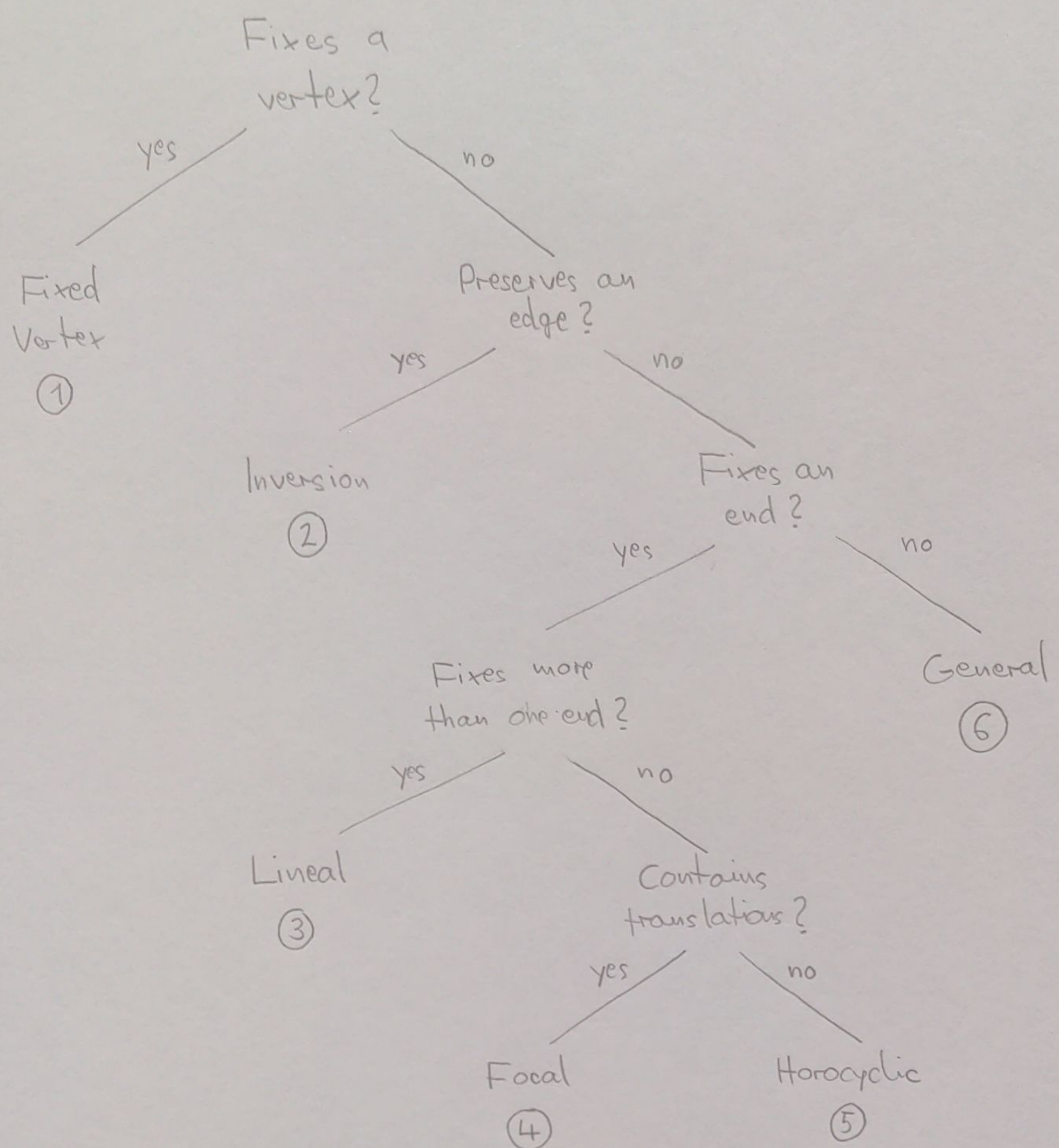
<u>Powerful correspondence between properties of $U(\Delta)$ and Δ</u>

1. $\{$ Fixed ends and invariant subtrees of $U(\Delta)\}$
   $\xleftrightarrow{1:1}$ $\{$ strongly confluent partial orientations of $\Delta \}$
   "scopo"

   (note: combinatorial in nature, computable when Δ is finite – GAP)

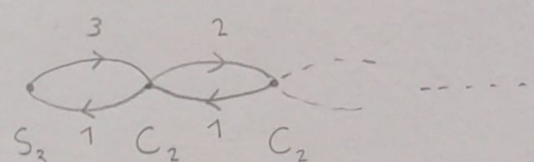
2. Local compactness, compact generation of $U(\Delta)$ $\leftrightarrow$ condition on Δ.
3. Action type of $U(\Delta)$ $\longleftrightarrow$ condition on Δ.
4. Discreteness of $U(\Delta)$ $\longleftrightarrow$ condition on Δ.

Six types of groups acting on trees



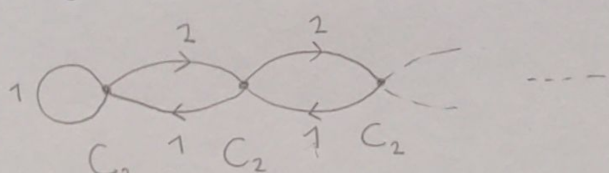
Examples for T_3 and general statement

1. $\mathrm{Aut}(T_3)_v$



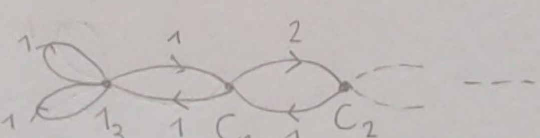
Γ is a tree and contains a single vertex cotree

2. $\mathrm{Aut}(T_3)_{\{a,\bar{a}\}}$



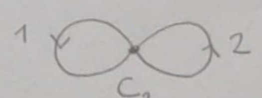
Γ contains a cotree consisting of a vertex with a non-orientable loop labelled by a set of size 1

3. $\mathrm{Aut}(T_3)_{\omega,\omega'}$



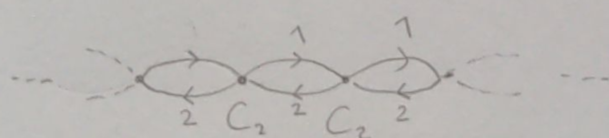
Γ contains a cyclic cotree all of whose arcs are labelled by a set of size 1

4. $\mathrm{Aut}(T_3)_\omega \cong \mathbb{Z} \ltimes H$



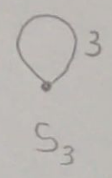
Γ contains a cyclic cotree with exactly one cyclic orientation in which all arcs have size 1 labels

5. H



Γ contains a unique horocyclic end

6. $\mathrm{Aut}(T_3)$



none of the above

Discreteness

Lem. Let T be a tree and $G \leq \mathrm{Aut}(T)$. Then G is discrete (in the permutation topology of $G \curvearrowright V(T)$) if and only if there is a finite set $F \subseteq V(T)$ such that G_F is trivial.

Thm. (Chijoff – T. '23) Let $\Delta = (\Gamma, (X_a), (G(v)))$ be a local action diagram. If $G := \mathcal{U}(\Delta)$ is of type

(Fixed Vertex) then G is discrete if and only if $G(v)$ is trivial for almost all $v \in V(\Gamma)$ and whenever X_v $(v \in V(\Gamma))$ is infinite then $G(v)$ has a finite base and $G(u)$ is trivial for every $u \in V(\Gamma)$ such that the arc $a \in o^{-1}(v)$ pointing towards u has an infinite colour set

(Inversion) then G is discrete if and only if --- (same as above)

(Lineal) then G is discrete if and only if $G(v)$ is trivial for all $v \in V(\Gamma)$

(Focal) then G is non-discrete

(Horocyclic) then G is non-discrete

(General) then there is a unique minimal cotree Γ' in Γ and G is discrete if and only if $G(v)$ is semiregular for all $v \in V(\Gamma')$ and trivial otherwise.