Discrete (P)-closed groups acting on trees

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(joint work with Marcus Chijoff)

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Idea

Classify all closed subgroups of $\mathrm{Aut}(T)$ by classifying all groups that can appear as $H^{(P_k)}$, i.e. all (P_k)-closed groups, and forming all intersections.

Whiteboard

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   - Reid '23: towards weakening the alternating assumption above

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Theorem (Reid–Smith '20)

$$\left\{ \begin{array}{c} \textit{Pairs } (G,T) \\ G \leq \mathrm{Aut}(T) \ \ \textit{is} \ \ (P_1)\textit{-closed} \end{array} \right\} / \cong \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \textit{local action} \\ \textit{diagrams} \end{array} \right\} / \cong$$

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Poster(s)

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A GAP package on local action diagrams is work in progress.
(joint with Marcus Chijoff)

Six types of groups acting on trees

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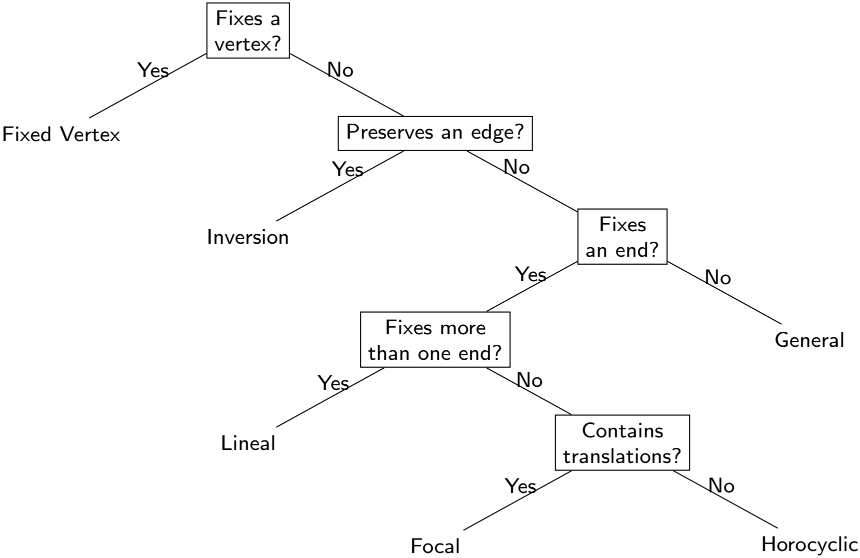

Whiteboard

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Inversion *if and only if* Δ *has a cotree consisting of a vertex with a self-reverse loop* $a \in A\Gamma$ *so that* $|X_a| = 1$.

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- Focal *if and only if Δ has a cyclic cotree Γ' with a cyclic orientation $O \subseteq A\Gamma'$ so that $|X_a| = 1$ for all $a \in O$ but there is an $a \in A(\Gamma') \setminus O$ with $|X_a| \geq 2$.*

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- **General** *if and only if none of the above apply.*

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- **Focal** *then it is non-discrete.*

Discrete (P)-closed groups

Theorem (Chijoff-T. '23)

Let $\Delta = (\Gamma, (X_a), (G(v)))$ be a local action diagram. If $U(\Delta)$ is of type

- *Fixed vertex* — *then it is discrete if and only if $G(v)$ is trivial for almost all $v \in V\Gamma$, and whenever X_v ($v \in V\Gamma$) is infinite then $G(v)$ has a finite base and $G(u)$ is trivial for every $u \in V\Gamma$ such that the arc $a \in o^{-1}(v)$ pointing towards u satisfies $|X_a| = \infty$.*
- *Inversion* — *then it is discrete if and only if ... (same as Fixed vertex).*
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- *Focal* — *then it is non-discrete.*
- *Horocyclic* — *then it is non-discrete.*

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- *Focal* then it is non-discrete.
- *Horocyclic* then it is non-discrete.
- *General* then it is discrete if and only if $G(v)$ is semiregular for all $v \in V\Gamma'$ and trivial otherwise; here Γ' is the unique smallest cotree of Δ.

The End.

Questions or comments?