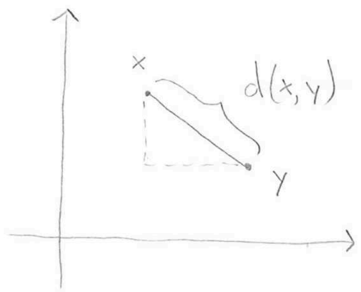
The Picard-Lindelöf theorem

(BMath Meetup, 13/03/24)

... one of my favourite theorems from one of my least favourite subjects.

Illustrates how a simple idea becomes powerful in abstraction and an interesting proof technique

Measuring distance



Can we measure the distance of points in sets other than $\mathbb{R}^n$?

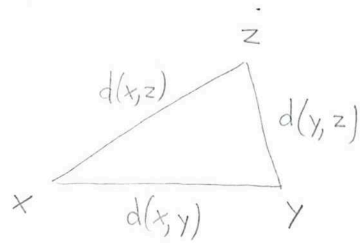
What properties would we expect "distance" to have?

Def. Let X be a set. A distance function on X, or metric, is a map $d: X \times X \longrightarrow \mathbb{R}_{\geq 0}$ such that

(i) $d(x,y) = 0 \iff x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$



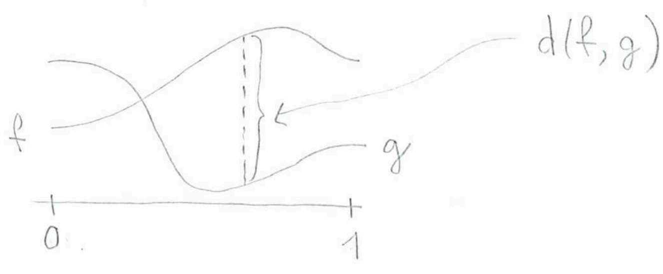
Example 1

$X = \mathbb{R}^2$, $d(x,y) :=$ "the usual distance"

Example 2

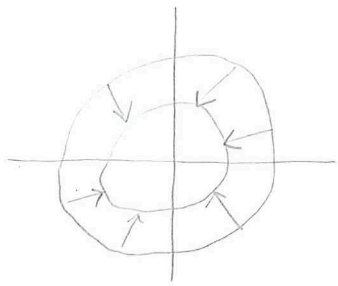
$X = \{\text{continuous functions from } [0,1] \text{ to } \mathbb{R}\} =: C([0,1], \mathbb{R})$

$d(f,g) := \sup\{|f(x)-g(x)| : x \in [0,1]\}$ this has all the properties!



Def. Let (X,d) be a metric space. A contraction of X is a map $c: X \to X$ such that there is $0 \leq \lambda < 1$ so that for all $x, y \in X$ we have $d(c(x), c(y)) \leq \lambda\, d(x,y)$.

Example $X = \mathbb{R}^2$ with the usual distance and $c(x) := \frac{1}{2} x$:



any two points move closer to each other (and towards the centre)

$$d(c(x), c(y)) = d\left(\tfrac{1}{2}x, \tfrac{1}{2}y\right) = \tfrac{1}{2} d(x,y)$$

imprecise ↓

Theorem (Banach 1922)

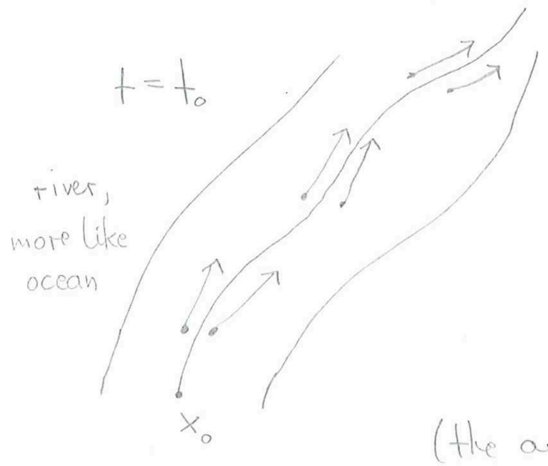
Every contraction has a unique fixed point.

Idea: the fixed point can be found by starting anywhere and repeatedly applying the contraction.

Uniqueness: if x, y are fixed points: $d(x,y) = d(c(x), c(y)) \leq \lambda\, d(x,y) \Rightarrow d(x,y) = 0 \overset{(i)}{\Rightarrow} x = y$.

And now: something completely different (?)

Let $F: \underset{\text{time}}{\mathbb{R}} \times \underset{\text{location}}{\mathbb{R}^n} \longrightarrow \underset{\text{vector}}{\mathbb{R}^n}$ be a time-dependent vector field



Given $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, is there a function $x: \mathbb{R} \longrightarrow \mathbb{R}^n$ such that

(i) $x(t_0) = x_0$

(ii) $x'(t) = F(t, x(t))$

(the answer should be yes – just drop a pebble into the river and see what happens)

Theorem (Picard–Lindelöf)

Yes, at least for a small time frame around the initial time point $x: (t_0 - \delta, t_0 + \delta) \longrightarrow \mathbb{R}^n$. (set $I_\delta := (t_0 - \delta, t_0 + \delta)$)

Proof (idea)

We need a function $x(t)$ so that $x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))\, ds$.

Then: $x(t_0) = x_0 + \int_{t_0}^{t_0} F(s, x(s))\, ds = x_0$ and

$$x'(t) = F(t, x(t)) \qquad \text{(fundamental theorem of calculus)}$$

We can't use this as a definition, however, because it is recursive. Instead, consider the metric space $X := (C(I_\delta, \mathbb{R}^n), d)$ and the map

$$c : X \longrightarrow X, \qquad x \longmapsto c(x)(t) := x_0 + \int_{t_0}^{t} F(s, x(s))\, ds.$$

We are looking for a fixed point of the map c.

Luckily, under some assumptions on F, the map c is a contraction:

$$d(c(x_1), c(x_2)) = \sup_{t \in I_\delta} \underbrace{\| c(x_1)(t) - c(x_2)(t) \|}$$

$$= \left\| \int_{t_0}^{t} F(s, x_1(s)) - F(s, x_2(s))\, ds \right\|$$

$$\le \int_{t_0}^{t} \| F(s, x_1(s)) - F(s, x_2(s)) \|\, ds$$

reasonable assumption: if locations are close then vectors are close

$$\le \int_{t_0}^{t} L \underbrace{\| x_1(s) - x_2(s) \|}_{\le d(x_1, x_2)}\, ds$$

$$\le \underbrace{\delta \cdot L}_{\text{small}} \cdot d(x_1, x_2)$$

So by Banach's theorem there is a fixed point. $\square$