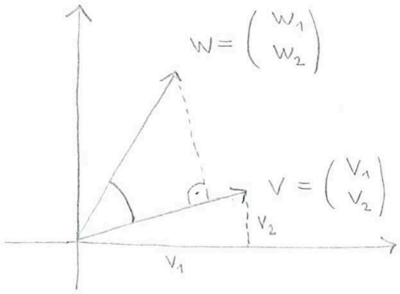
Fourier series

(BMath Meetup, 27/03/24, ≈ 45 minutes)

Similar spirit to Picard-Lindelöf talk:
start with a simple geometric idea, make it more abstract, and then apply it in an unexpected setting

Simple idea: measuring angles



the angle can be computed using an appropriate right-angled triangle

This can be done algebraically in terms of the coordinates, which is neat in itself:

Define $\langle v, w \rangle := v_1 w_1 + v_2 w_2$. Then the length of a vector v is $\|v\| := \sqrt{\langle v, v\rangle}$ and one obtains that

$$\langle v, w \rangle = \|v\| \cdot \|w\| \cdot \cos(\measuredangle(v,w))$$

For example, take $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The angle between them is $90°$, so $\cos(\measuredangle(v,w)) = 0$.

Indeed we compute: $\langle v, w \rangle = 1 \cdot (-1) + 1 \cdot 1 = 0$.

So whenever v, w are non-zero vectors then $\measuredangle(v,w) = 90°$ (v, w are orthogonal) if and only if $\langle v, w\rangle = 0$ and the length of v is 1 if and only if $\langle v, v \rangle = 1$.

This remains true for vectors in $\mathbb{R}^n$, for example:

$$v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad ; \quad \langle v, w \rangle = 1 \cdot (-1) + 0 \cdot 2 + 1 \cdot 1 = 0$$

$\rightsquigarrow$ there is a right angle between v and w

<u>Def.</u> A set of vectors $(v_1, \dots, v_n) \in \mathbb{R}^n$ is an orthonormal basis of $\mathbb{R}^n$ if $\langle v_i, v_j \rangle = 0$ for $i \neq j$ and $\langle v_i, v_i \rangle = 1$ for all i.

(pairwise orthogonal and length 1)

<u>Ex.</u> $\mathbb{R}^3$, take $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

A benefit of an orthonormal basis is that we can decompose any vector: for example:

$$w = \begin{pmatrix} 5 \\ 3 \\ -7 \end{pmatrix}, \quad w = \underset{\substack{\uparrow \\ =\langle w, v_1\rangle}}{5} \cdot \overset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + \underset{\substack{\uparrow \\ \langle w, v_2\rangle}}{3} \cdot \overset{v_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} - \underset{\substack{\uparrow \\ \langle w, v_3\rangle}}{7} \cdot \overset{v_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

why use a formula if we can read off the numbers directly?

Question: can we measure angles between more complicated objects, and decompose those objects

be able to add and scale elements

Def. Let V be a vector space. An inner product on V is a map $\langle -, - \rangle : V \times V \longrightarrow \mathbb{R}$ such that

(i) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \iff v = 0$

(the length of a vector is non-negative, and 0 if and only if the vector is 0)

(ii) $\langle v, w \rangle = \langle w, v \rangle$

(the angle between v and w is the same as the angle between w and v).

(iii) $\langle av + bw, u \rangle = a\langle v, u \rangle + b\langle w, u \rangle$

(computing angles for sums goes back to the summands)

Example 1 $\quad V = \mathbb{R}^n, \ \langle v, w \rangle = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$

Example 2 $\quad V = \{ 2\pi\text{-periodic functions } f \text{ from } \mathbb{R} \text{ to } \mathbb{R}, \text{ i.e. } f(x+2\pi) = f(x) \text{ for all } x \} = C_{2\pi}(\mathbb{R})$

note: this is a vector space: can add any two, or multiply (pointwise) by a number and still be 2π-periodic

Elements? $\sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x), \dots,$ constant

Can we make sense of an angle / inner product between two such functions? Similar to the $\mathbb{R}^n$ case, define for $f, g \in C_{2\pi}(\mathbb{R})$

$$\langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

For example:

$$\langle \sin(x), \cos(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \overbrace{\underbrace{\sin(x)}_{\text{odd}}\ \underbrace{\cos(x)}_{\text{even}}}^{\text{odd}}\, dx = 0$$

$$\langle \sin(x), \sin(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(x)\, dx = \ldots \text{trigonometric identity} \ldots = 1$$

Actually:

$$\langle \sin(nx), \sin(mx) \rangle = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases} = \langle \cos(nx), \cos(mx) \rangle$$

$$\langle \sin(nx), \cos(mx) \rangle = 0$$

Do we get an orthonormal basis? Is it true that any $f \in C_{2\pi}(\mathbb{R})$ can be written as

$$f(x) = \sum_{n=0}^{\infty} \Big(\underbrace{\langle f, \sin(nx) \rangle}_{} \cdot \sin(nx) + \underbrace{\langle f, \cos(nx) \rangle}_{} \cdot \cos(nx) \Big) \quad ?$$

coefficients as before

Cut it off after finitely many terms to get an approximation?
$\rightsquigarrow$ picture / gif from wikipedia