Discrete (P)-closed groups acting on trees

Stephan Tornier
(joint work with Marcus Chijoff)

THE UNIVERSITY OF
NEWCASTLE
AUSTRALIA

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Idea

Classify all closed subgroups of $\mathrm{Aut}(T)$ by classifying all groups that can appear as $H^{(P_k)}$, i.e. all (P_k)-closed groups, and forming all intersections.

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   - Reid '23: towards weakening the alternating assumption above

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Theorem (Reid–Smith '20)

$$\left\{ \begin{array}{c} \textit{Pairs } (G,T) \\ G \leq \mathrm{Aut}(T) \;\; \textit{is} \;\; (P_1)\textit{-closed} \end{array} \right\} / \cong \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \textit{local action} \\ \textit{diagrams} \end{array} \right\} / \cong$$

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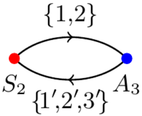

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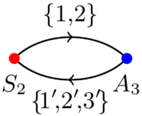

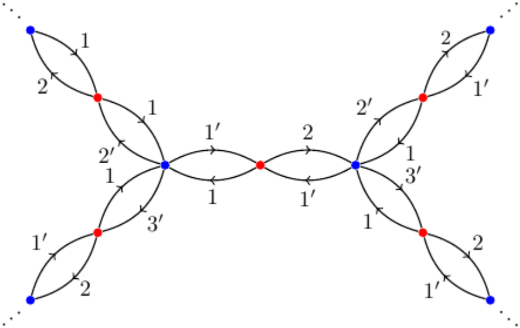

Poster(s)

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A GAP package on local action diagrams is work in progress.
(joint with Marcus Chijoff)

Six types of groups acting on trees

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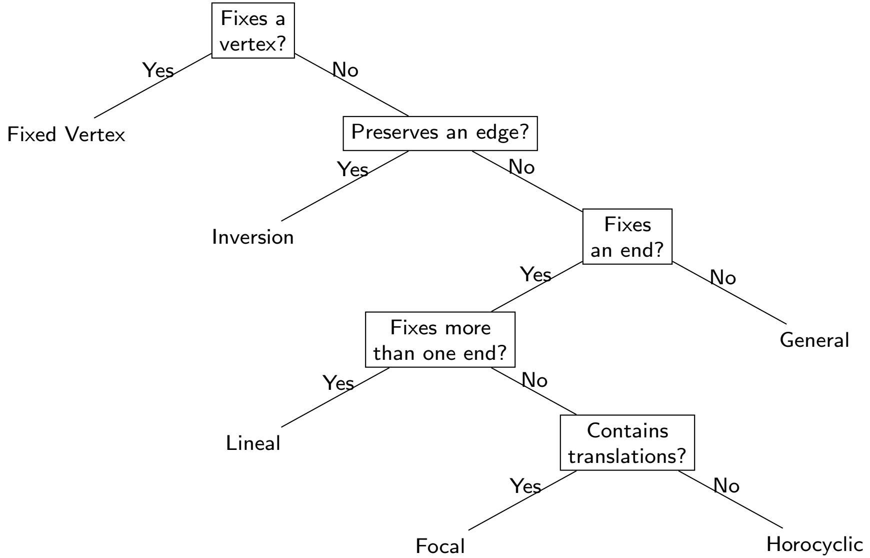

Blackboard

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- **Focal** *if and only if Δ has a cyclic cotree Γ' with a cyclic orientation $O \subseteq A\Gamma'$ so that $|X_a| = 1$ for all $a \in O$ but there is an $a \in A(\Gamma') \setminus O$ with $|X_a| \geq 2$.*

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- **Horocyclic** *if and only if Γ is a tree and Δ has a unique horocyclic end.*

Six types of groups acting on trees

Theorem

Let $\Delta = (\Gamma, (G(v)), (X_a))$ be a local action diagram. Then $U(\Delta)$ is of type

- **Fixed vertex** *if and only if Γ is a tree and Δ has a single vertex cotree.*
- **Inversion** *if and only if Δ has a cotree consisting of a vertex with a self-reverse loop $a \in A\Gamma$ so that $|X_a| = 1$.*
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- **Horocyclic** *if and only if Γ is a tree and Δ has a unique horocyclic end.*
- **General** *if and only if none of the above apply.*

Blackboard

Discrete (P)-closed groups

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Theorem (Chijoff-T. '24)

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Discrete (P)-closed groups

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Fixed vertex *then it is discrete if and only if $G(v)$ is trivial for almost all $v \in V\Gamma$, and whenever X_v ($v \in V\Gamma$) is infinite then $G(v)$ has a finite base and $G(u)$ is trivial for every $u \in V\Gamma$ such that the arc $a \in o^{-1}(v)$ pointing towards u satisfies $|X_a| = \infty$.*

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- **Focal** *then it is non-discrete.*
- **Horocyclic** *then it is non-discrete.*
- **General** *then it is discrete if and only if $G(v)$ is semiregular for all $v \in V\Gamma'$ and trivial otherwise; here Γ' is the unique smallest cotree of Δ.*

The End.

Questions or comments?