Recent developments in groups acting on trees

Stephan Tornier
(joint work with Marcus Chijoff)

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

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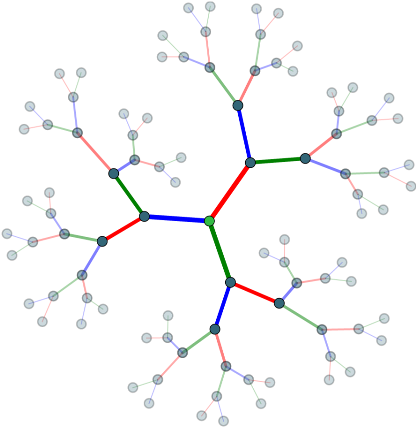
Introduction



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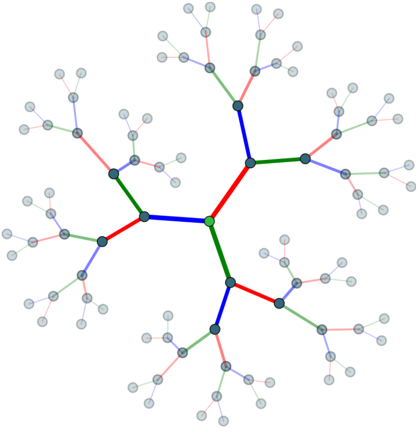

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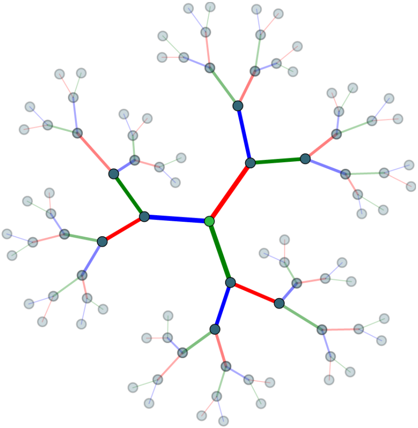

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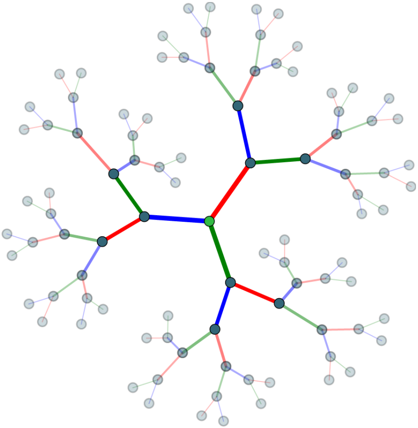

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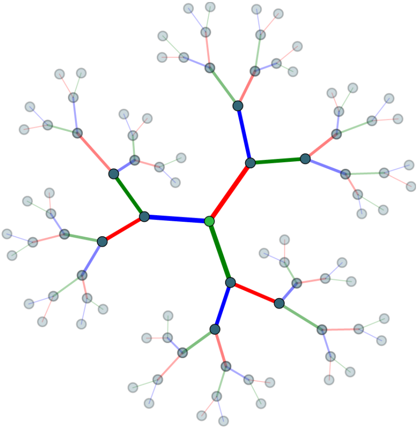

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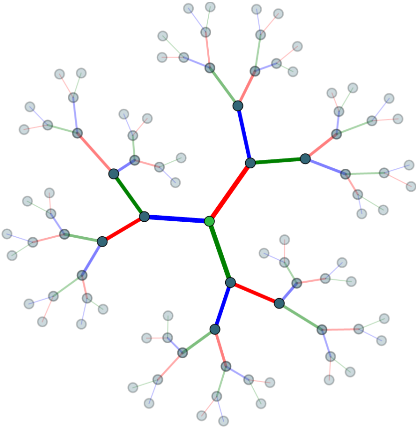

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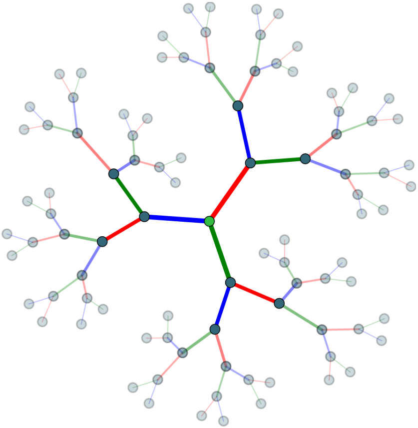

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A subgroup $H \leq \mathrm{Aut}(T)$ is discrete if and only if $H_S = \{\mathrm{id}\}$ for a finite S.

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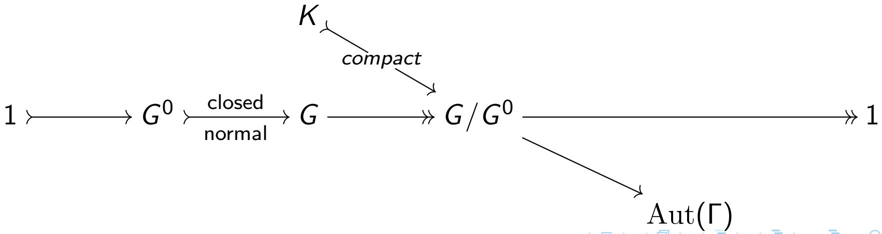

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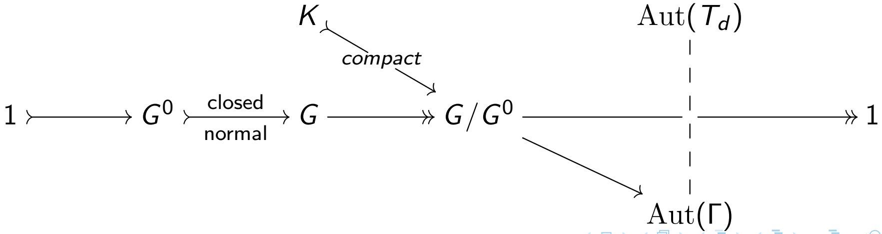

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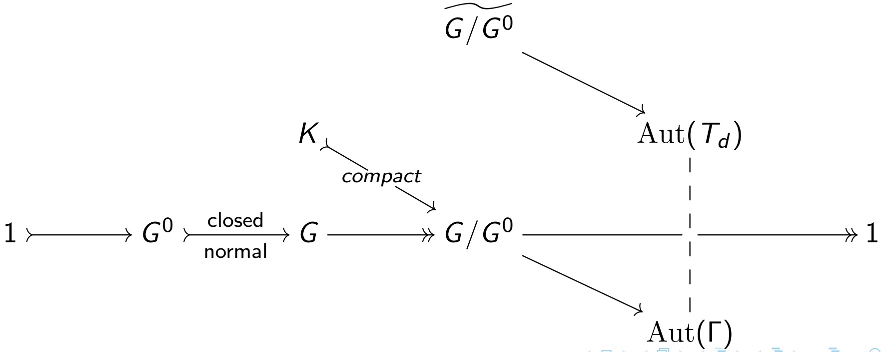

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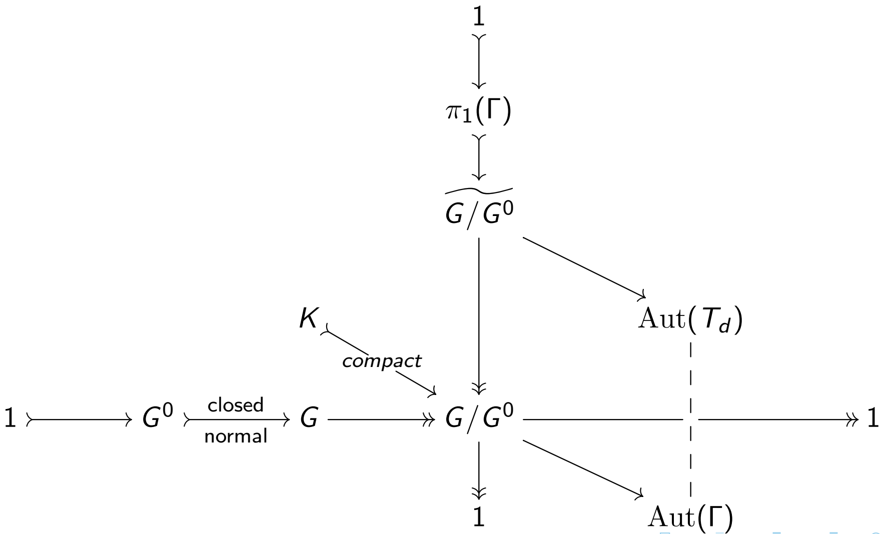

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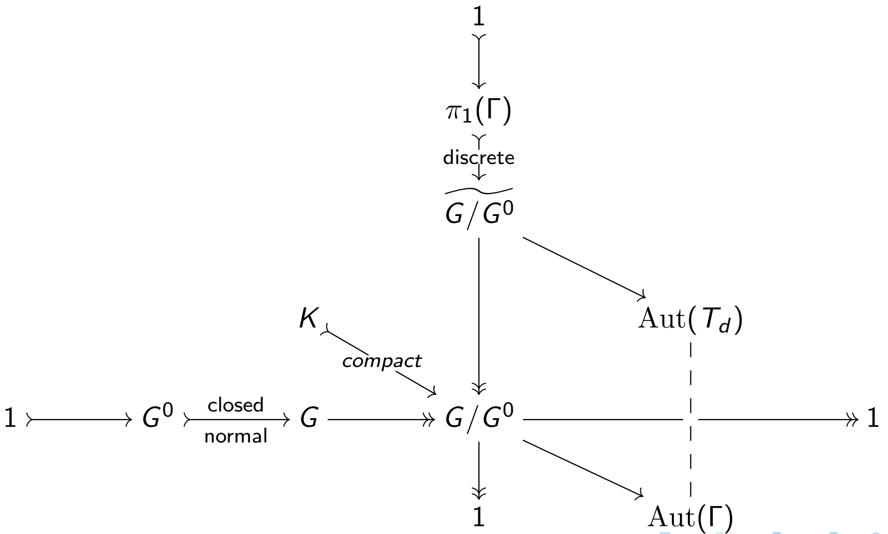

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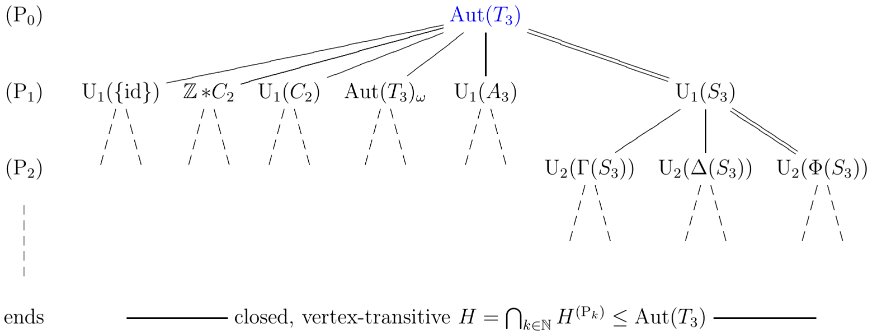
Idea

Classify all closed subgroups of $\mathrm{Aut}(T)$ by classifying all groups that can appear as $H^{(P_k)}$, i.e. all (P_k)-closed groups, and forming all intersections.

Towards a classification of closed vertex-transitive groups



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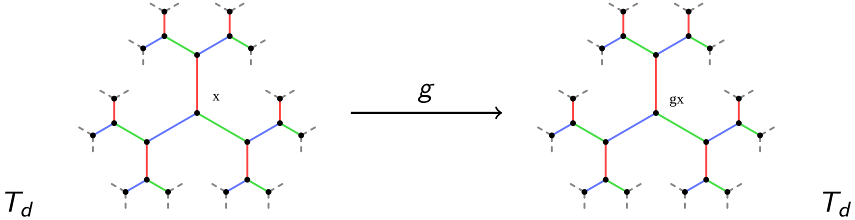

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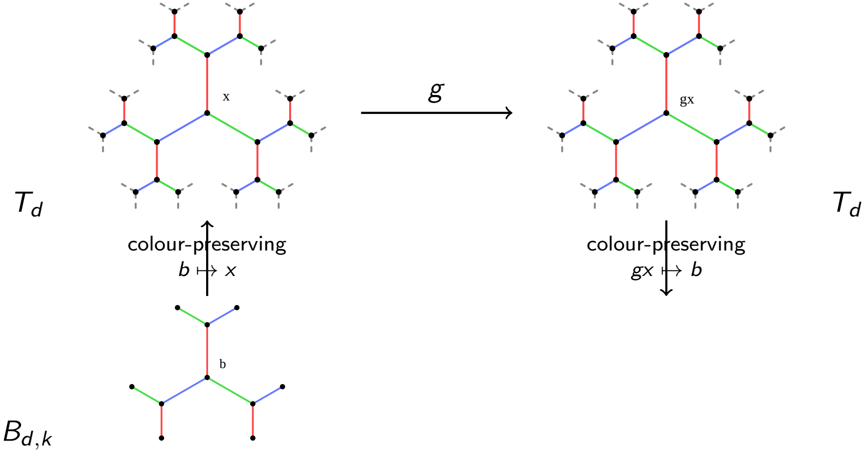

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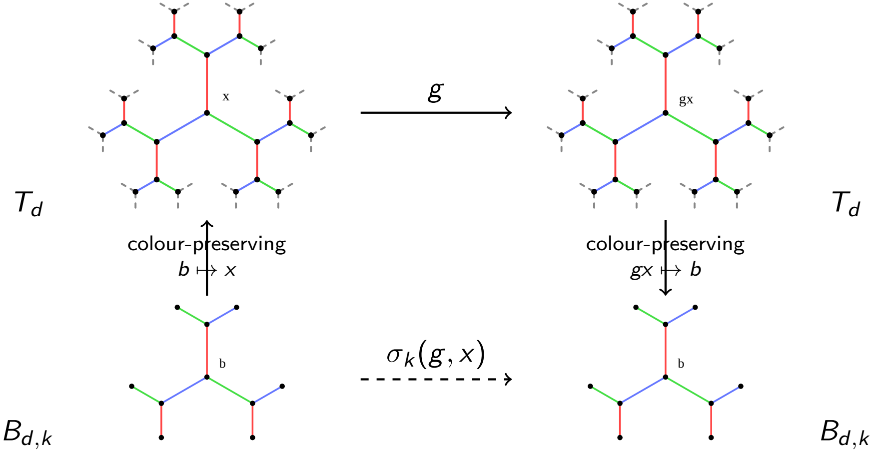

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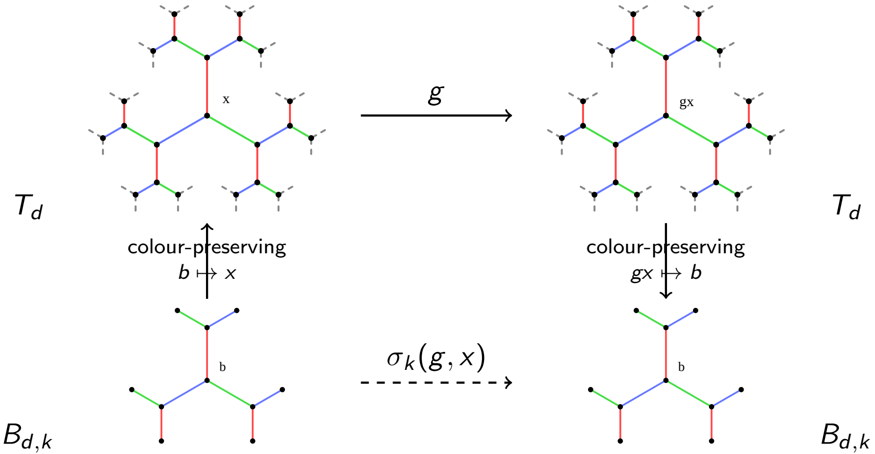

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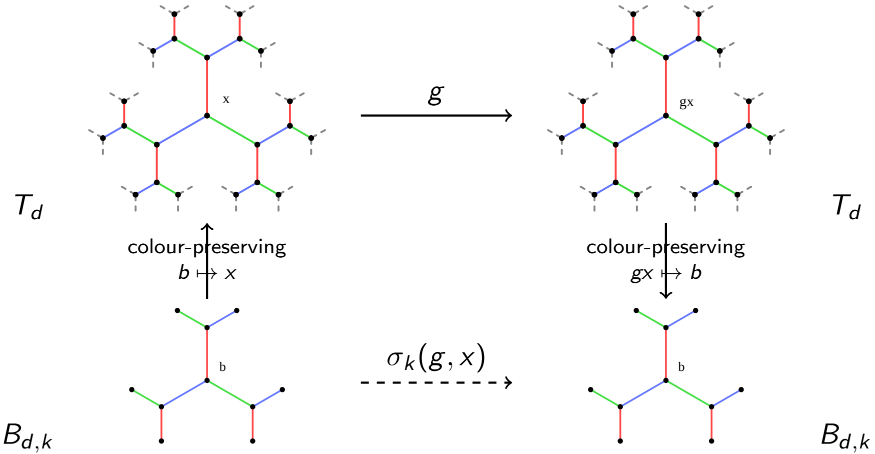

Definition

For $F \leq \mathrm{Aut}(B_{d,k})$, set $\mathrm{U}_k(F) := \{g \in \mathrm{Aut}(T_d) \mid \forall x \in V(T_d) : \sigma_k(g, x) \in F\}$.

Classification results/plans

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Classification results/plans

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   - Reid '23: towards weakening the alternating assumption above

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Theorem (Reid–Smith '20)

$$\left\{ \begin{array}{c} \textit{Pairs } (G,T) \\ G \leq \mathrm{Aut}(T) \ \textit{ is } \ (P_1)\textit{-closed} \end{array} \right\} / \cong \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \textit{local action} \\ \textit{diagrams} \end{array} \right\} / \cong$$

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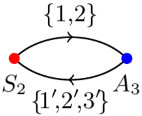

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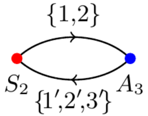

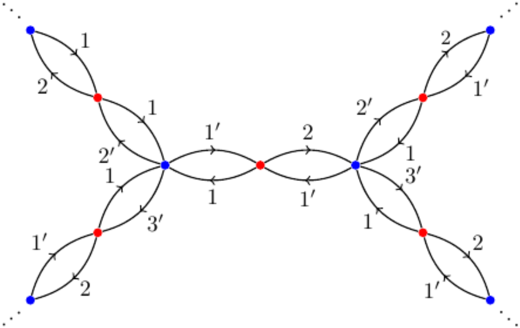

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A GAP package on local action diagrams is work in progress.
(joint with Marcus Chijoff)

Six types of groups acting on trees

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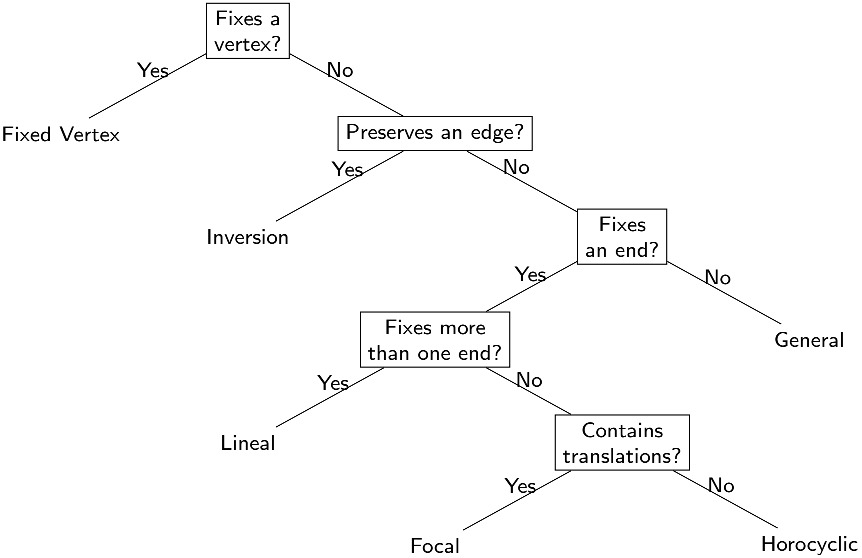
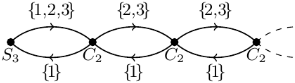
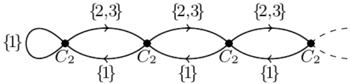
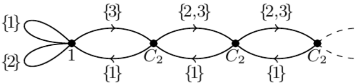
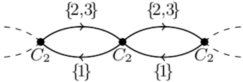
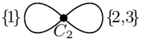
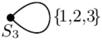


Type	G	$\Delta(T, G)$
(*Fixed Vertex*)	$\mathrm{Aut}(T)_x$	
(*Inversion*)	$\mathrm{Aut}(T)_{\{a,\overline{a}\}}$	
(*Lineal*)	$\mathrm{Aut}(T)_{\omega,\omega'}$	
(*Horocyclic*)	H	
(*Focal*)	$\mathrm{Aut}(T)_\omega$	
(*General*)	$\mathrm{Aut}(T)$	

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- **General** *if and only if none of the above apply.*

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Fixed vertex *then it is discrete if and only if $G(v)$ is trivial for almost all $v \in V\Gamma$, and whenever X_v $(v \in V\Gamma)$ is infinite then $G(v)$ has a finite base and $G(u)$ is trivial for every $u \in V\Gamma$ such that the arc $a \in o^{-1}(v)$ pointing towards u satisfies $|X_a| = \infty$.*

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- **Focal** *then it is non-discrete.*
- **Horocyclic** *then it is non-discrete.*
- **General** *then it is discrete if and only if $G(v)$ is semiregular for all $v \in V\Gamma'$ and trivial otherwise; here Γ' is the unique smallest cotree of Δ.*

The End.
Questions or comments?