AN INTRODUCTION TO GROUPS ACTING ON TREES

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Abstract. This document is an extended version of lecture notes for a three-hour minicourse delivered by the author at the "Groups in the Midlands" workshop in Lincoln, UK during the week June 17-20, 2025.

First, we motivate the study of groups acting on trees through the Cayley–Abels graphs construction and the associated action map. Secondly, we organise groups acting on trees into six different types, and introduce independence as well as transitivity properties. Finally, we introduce generalised universal groups and identify them as those locally transitive groups acting on trees that satisfy an independence property and contain an involutive edge inversion.

1. Cayley–Abels graphs

In the general theory of totally disconnected, locally compact (t.d.l.c.) groups, groups acting on trees play an important role for theoretical and practical reasons. The theoretical side is summarised by Figure 1. We elaborate on it below.

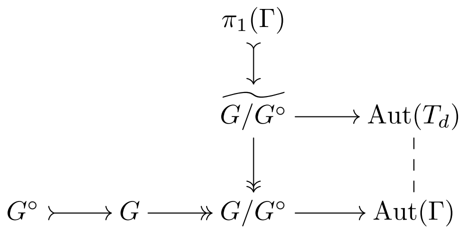

Figure 1. Groups acting on trees among locally compact groups.

Let G be a locally compact group. The connected component of the identity in G is a closed normal subgroup, which is locally compact and connected. Such groups are known to be inverse limits of Lie groups by the solution to Hilbert's fifth problem due to [Gle52], [MZ52] and [Yam53] in the 1950s. The quotient group G/G° is locally compact and totally disconnected. This class of groups is less well understood and has received much attention in recent years.

Example 1.1. Connected versus totally disconnected locally compact groups.

(i) Examples of connected locally compact groups include the real numbers $(\mathbb{R}, +)$, the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, the group of invertible matrices of positive determinant $\mathrm{GL}_+(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) > 0\}$ and the special linear group $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) = 1\}$ $(n \in \mathbb{N})$.

(ii) Examples of totally disconnected locally compact groups include abstract groups equipped with the discrete topology, profinite groups, the p-adic numbers $(\mathbb{Q}_p, +)$, the general linear group $\mathrm{GL}(n, \mathbb{Q}_p)$ $(n \in \mathbb{N})$, automorphism groups of locally finite graphs and buildings, and Kac-Moody groups.

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We remark that the smallest class of second countable t.d.l.c. groups that contains all discrete groups and profinite such groups, and is closed under group extensions as well as countable directed unions is known as the class of *elementary* t.d.l.c. groups, after Wesolek [Wes15], in analogy to elementary amenable groups.

Example 1.2. Examples of the sequence $G^\circ \rightarrowtail G \twoheadrightarrow G/G^\circ$ of Figure 1.

(i) Let $G := \mathrm{GL}(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\}$. Then $G^\circ = \mathrm{GL}_+(n,\mathbb{R})$ and we have $G/G^\circ \cong \mathbb{Z}/2\mathbb{Z}$.
(ii) Let $G := \mathrm{O}(1,1) = \{A \in \mathbb{R}^{n\times n} \mid B_{1,1}(Ax,Ay) = B_{1,1}(x,y)\ \forall x,y \in \mathbb{R}^2\}$, where $B_{1,1}$ is the bilinear form $B_{1,1}((x_1,y_1)^T,(x_2,y_2)^T) := x_1x_2 - y_1y_2$. The group $\mathrm{O}(1,1)$ preserves the two-sheeted hyperbola $x^2 - y^2 = -1$ and its connected component $G^\circ = \mathrm{SO}^+(1,1)$ is the subgroup of $\mathrm{O}(1,1)$ consisting of matrices of positive determinant that preserve each of the two sheets. One can show that $G/G^\circ \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
(iii) The automorphism group $G := \mathrm{Aut}(T_d)$ of the d-regular tree ($d \in \mathbb{N}_{\geq 3}$), carrying the permutation topology for its action on the vertex set VT_d, is locally compact and totally disconnected. In particular $G^\circ = \{\mathrm{id}\}$ and $G/G^\circ \cong G$.
(iv) Let $G := \mathrm{GL}(n,\mathbb{R}) \times \mathrm{Aut}(T_d)$. Then $G^\circ = \mathrm{GL}_+(n,\mathbb{R}) \times \{\mathrm{id}\}$ and we have $G/G^\circ \cong \mathbb{Z}/2\mathbb{Z} \times \mathrm{Aut}(T_d)$. In general, G is a group extension of G° by G/G°.

By a result of Abels [Abe74], *compactly generated* t.d.l.c. groups always admit a nice action on a locally finite connected graph, generalising the action of a finitely generated group on a Cayley graph. See also Krön–Möller [KM08], and the survey articles by Lederle [Led22] and Wesolek [Wes18], which we follow closely.

Definition 1.3. Let G be a t.d.l.c. group. A **Cayley–Abels** graph for G is a locally finite connected graph Γ on which G acts vertex-transitively with compact open vertex stabilisers.

Note that due to the vertex-transitivity assumption a Cayley–Abels graph is necessarily regular. The following three examples are immediate from the definition.

Example 1.4. Cayley–Abels graphs.

(i) Let $G = \langle S \mid R\rangle$ be a finitely generated group with the discrete topology. Then the Cayley graph $\Gamma := \mathrm{Cay}(G,S)$ with vertex set $V\Gamma := G$ and edge set $E\Gamma := \{\{g,gs\} \mid g \in G,\ s \in S\}$ satisfies Definition 1.3. It is locally finite because S is finite and connected because G is generated by S. Moreover, the group G acts vertex-transitively by left multiplication on $V\Gamma = G$, and all vertex-stabilisers coincide with the trivial subgroup of G, which is compact and open in the discrete topology.
(ii) Let G be a compact group. Then we may take Γ to consist of a single vertex, which is fixed by G. Note that the kernel of this action is all of G, so all profinite groups become trivial in this sense.
(iii) Let $G := \mathrm{Aut}(T_d)$ for some $d \in \mathbb{N}$. Then we may take $\Gamma := T_d$.

We show that the existence of a Cayley–Abels graph implies compact generation. The proof exhibits a finitely generated, vertex-transitive subgroup.

Proposition 1.5. Let G be a t.d.l.c. group, and let Γ be a Cayley–Abels graph for G. Then G is compactly generated.

Proof. Let $v \in V\Gamma$. Then the stabiliser G_v is a compact open subgroup of G. It will be a subset of the compact generating set we construct.

As Γ is locally finite, there are only finitely many neighbours $\{v_1,\dots,v_n\} \subseteq V\Gamma$ of $v \in V\Gamma$. By vertex-transitivity, there are elements $g_1,\dots,g_n \in G$ such that $g_iv = v_i$ ($i \in \{1,\dots,n\}$). We claim that G is generated by the compact set $G_v \cup \{g_1,\dots,g_n\}$.

It suffices to prove that $D := \langle g_1, \ldots, g_n \rangle$ acts vertex-transitively on Γ: if, given $g \in G$, there is $d \in D$ such that $gv = dv$ then $d^{-1}g \in G_v$ and hence $g \in DG_v$.

To see that D acts vertex-transitively, we argue by induction on $k \in \mathbb{N}$ that for all $w \in B(v,k)$ there is $d \in D$ such that $dv = w$. Since Γ is connected, every vertex of Γ lies in $B(v,k)$ for some $k \in \mathbb{N}$. By definition of D, the statement holds for $k = 1$. Suppose that the statement holds for $k \in \mathbb{N}$ and let $w \in V\Gamma$ be such that $d(v,w) = k+1$. Let $(v, u_1, u_2, \ldots, u_k, w)$ be a geodesic from v to w, see Figure 2.

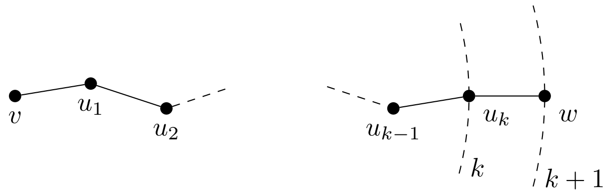

FIGURE 2. A geodesic connecting v to w.

Then $u_1 \in \{v_1, v_2, \ldots, v_n\}$. Using the induction hypothesis, let $d \in D$ be so that $du_k = v$. Then $dw = v_i$ for some $i \in \{1, \ldots, n\}$ and $g_i^{-1}dw = v$, with $g_i^{-1}d \in D$. □

Conversely, compact generation guarantees the existence of a Cayley–Abels graph. Its vertices can be chosen to be the left cosets of any compact open subgroup.

Theorem 1.6. Let G be a compactly generated t.d.l.c. group. Then G admits a Cayley–Abels graph Γ. Given any compact open subgroup U of G, we may choose $V\Gamma = G/U$ on which G acts by left multiplication.

First, using compact generation of G, we construct a finite set $A \subseteq G$ that plays the role of the finite generating set S in the context of classical Cayley graphs.

Lemma 1.7. Let G be a compactly generated t.d.l.c. group. Further, let U be a compact open subgroup of G and X a compact generating set of G. Then

(i) there is a finite symmetric set $A \subseteq G$ containing $1 \in G$ such that $X \subseteq AU$ and $UAU = AU$, and
(ii) for any finite symmetric set $A \subseteq G$ which satisfies $X \subseteq AU$ and $UAU = AU$, it is the case that $G = \langle A \rangle U$.

Proof. For part (i), note that $\{xU \mid x \in X\}$ is an open cover of X in G, so there is a finite symmetric set B containing 1 such that $X \subseteq BU$. On the other hand, the set UBU, which contains UB is compact as well, and covered by $\{ubU \mid u \in U, b \in B\}$. Hence there is a finite symmetric set A containing 1 such that $UB \subseteq AU$ and $B \subseteq A \subseteq UB$. As a consequence, $UAU \subseteq UBU \subseteq AUU = AU$.

For part (ii), note inductively that $(UAU)^n = A^nU$ $(n \in \mathbb{N})$. Since UAU contains X and is symmetric it follows that

$$G = \langle UAU \rangle = \bigcup\nolimits_{n\in\mathbb{N}} (UAU)^n = \bigcup\nolimits_{n\in\mathbb{N}} A^nU = \langle A \rangle U. \qquad \square$$

Equipped with Lemma 1.7, we now turn to the proof of Theorem 1.6.

Proof. (Theorem 1.6). Let A be as in Lemma 1.7, so $UAU = AU$ and $G = \langle A \rangle U$. Set $V\Gamma := G/U$ and consider the trivial coset $U \in G/U$. We define

$$E\Gamma := \{\{gU, gaU\} \mid g \in G,\ a \in A \backslash \{1\}\}.$$

By definition of $V\Gamma$, the group G acts vertex-transitively by left multiplication. The defintion of $E\Gamma$ shows that this action is by automorphisms of Γ. Moreover, all vertex stabilisers are conjugates of U, and hence compact open. It remains to show that Γ is connected and locally finite.

For connectivity, we use that $G = \langle A \rangle U$. Given $g \in G$, write $g = a_1 \cdots a_n u$. Then

$$U, a_1 U, a_1 a_2 U, \ldots, a_1 \cdots a_n U = a_1 \ldots a_n u U$$

is a path in Γ connecting U and gU.

For local finiteness, it suffices to show that $B(U,1) = \{aU \mid a \in A\}$: since G acts vertex-transitively this implies that Γ is regular of degree $|A|$. Let $\{gU, gaU\} \in E\Gamma$, and suppose that either $gU = U$ or $gaU = U$.

If $gU = U$ then $g \in U$. Hence $gaU \in UAU = AU$, and so we conclude that $gaU = a'U$ for some $a' \in A$ as desired.

If $gaU = U$ then $g = ua^{-1}$ for some $u \in U$. Therefore we have $gU \in UAU = AU$, considering that A is symmetric, Consequently, $g = a'u'$ for some $a' \in A$ and $u' \in U$. That is, $gU = a'U$ as desired. □

Given a compactly generated t.d.l.c. group G and a Cayley–Abels graph Γ of G, we now collect various useful properties of the action map $G \to \mathrm{Aut}(\Gamma)$.

Proposition 1.8. Let G be a t.d.l.c. group and Γ be a Cayley–Abels graph of G. Then the map $\varphi : G \to \mathrm{Aut}(\Gamma)$ is continuous and closed. Moreover, $\ker(\varphi) \subseteq G$ is compact and $\varphi(G)$ is cocompact in $\mathrm{Aut}(\Gamma)$.

Proof. For continuity, consider a basic open set U of $\mathrm{Aut}(\Gamma)$, that is, a stabiliser of a finite set of vertices and edges. Since G acts with compact open vertex stabilisers, the preimage $\varphi^{-1}(U)$ is an intersection of finitely many open sets and hence open.

To see that φ is a closed map, consider a closed set $A \subseteq G$ and let $(a_i)_{i \in \mathbb{N}}$ be a sequence in A, so that $(\varphi(a_i))_{i \in \mathbb{N}}$ converges to an element $h \in \mathrm{Aut}(\Gamma)$. Fix $v \in V\Gamma$. Then there is $N \in \mathbb{N}$ such that for all $i, j \geq N$ we have $\varphi(a_i^{-1})\varphi(a_j) \in \mathrm{Aut}(\Gamma)_v$. Hence $a_i^{-1} a_j \in G_v$ for all $i, j \geq N$. In particular, $a_N^{-1} a_j \in G_v$ for all $j \geq N$. Since G_v is compact, we conclude that there is a convergent subsequence $(a_N^{-1} a_{j_k})_{k \in \mathbb{N}}$ of $(a_N^{-1} a_j)_{j \in \mathbb{N}}$ converging to some $b \in G_v$. Hence $(a_{j_k})_{k \in \mathbb{N}}$ converges, to some $a \in A$. Since φ is continuous we have $\varphi(a) = h$, and so φ is closed.

The kernel $\ker(\varphi) = \bigcap_{v \in V\Gamma} G_v$ is compact as an intersection of compact sets due to the fact that G acts with compact open vertex stabilisers.

Concerning the image, recall the proof of Proposition 1.5. Since $\varphi(G) \subseteq \mathrm{Aut}(\Gamma)$ is vertex-transitive, we have $\mathrm{Aut}(\Gamma) = \varphi(G) \cdot \mathrm{Aut}(\Gamma)_v$. Given that $\mathrm{Aut}(\Gamma)_v$ is compact, so is the quotient space $\varphi(G) \backslash \mathrm{Aut}(\Gamma)$. □

Cayley–Abels graphs are of theoretical importance and motivate the study of groups acting on graphs. For example, as a consequence of Theorems 1.6 and 1.8, every non-compact, compactly generated simple t.d.l.c. group can be viewed as a closed vertex-transtive subgroup of a locally finite regular connected graph: the kernel of the represenation map is necessarily trivial.

However, Cayley–Abels graphs are difficult to construct and analyse explicitly as the following two remarks illustrate.

Remark 1.9. Intricacies of Cayley–Abels graphs.

(i) We know that for $d \in \mathbb{N}$, the regular tree T_d is a Cayley–Abels graph for the group $\mathrm{Aut}(T_d)$. However, could there be a Cayley–Abels graph, not necessarily a tree, of smaller degree? This question is surprisingly subtle and was answered negatively in the research article [ÁLM23].

(ii) It is stated as a problem in [Led22, Problem 2.21] to give a good description of a Cayley–Abels graph for Neretin's group of tree almost automorphisms.

Finally, the action of a t.d.l.c. group G on a Cayley–Abels graph Γ lifts to an action of a group $\widetilde{G}$ on the universal cover of Γ, which is the regular tree of the same degree. The group $\widetilde{G}$ contains the fundamental group $\pi_1(\Gamma)$ as a discrete normal subgroup and $G \cong \widetilde{G}/\pi_1(\Gamma)$. A good account of this is given by [Led22, Section 3].

2. Independence and Transitivity Properties

This article is not self-contained in that it assumes basic terminology concerning trees and groups acting on them, see for example [GGT18]. Group actions on trees can be coarsely organised into six different types, according to invariant structures.

Proposition 2.1 ([Tit70], [RS20, Thm. 2.5])**.** Let G be a group acting on a tree T. Then G belongs to exactly one of the following types: the group G either

(*Fixed vertex*) fixes a (not necessarily unique) vertex,
(*Inversion*) preserves a unique edge and contains an inversion of that edge,
(*Lineal*) fixes exactly two ends and translates the line between them,
(*Focal*) fixes a unique end and contains a translation towards this end,
(*Horocyclic*) fixes a unique end but no vertices, and acts without translation, or
(*General*) acts with translation and does not fix any end.

Proposition 2.1 is summarised by the following decision tree of Figure 3.

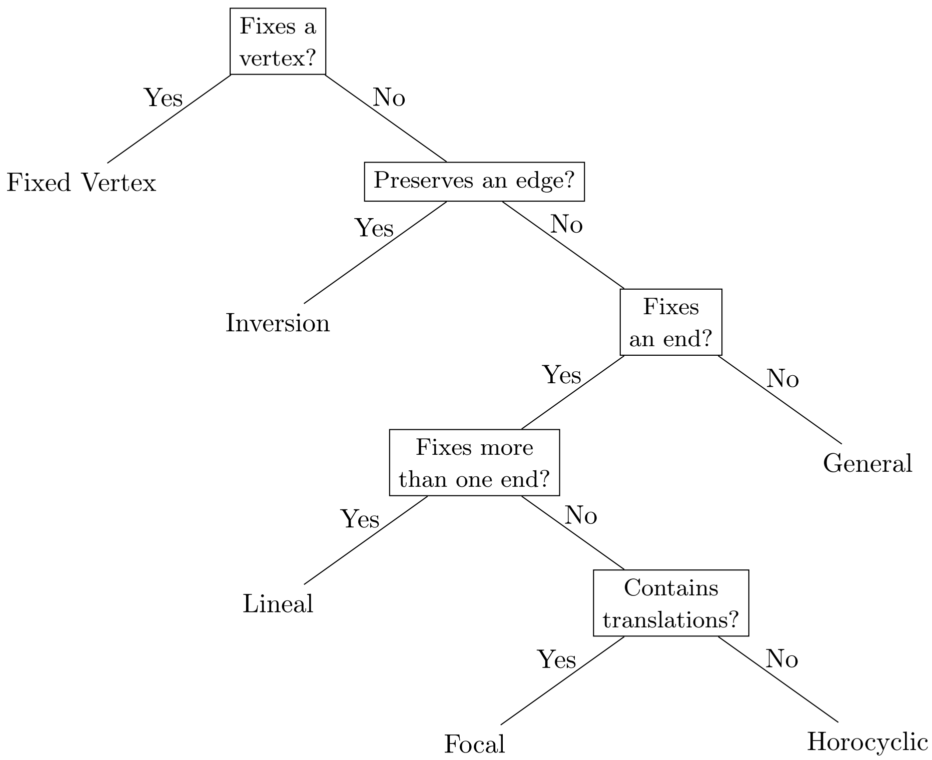

Figure 3. Decision tree of Proposition 2.1.

Example 2.2. We describe prototypical examples of groups of each type in the case of $T = T_3$ as follows. Let $v \in VT$, $e \in ET$ and $\omega, \omega' \in \partial T$. Recall that $\mathrm{Aut}(T)_\omega$ splits as a semidirect product $\mathrm{Aut}(T)_\omega \cong \mathbb{Z} \ltimes H$, where H consists of all elliptic elements in $\mathrm{Aut}(T)_\omega$ and $\mathbb{Z}$ is generated by a translation towards ω.

Fixed vertex	Inversion	Lineal	Focal	Horocyclic	General
$\mathrm{Aut}(T)_v$	$\mathrm{Aut}(T)_{\{e,\overline{e}\}}$	$\mathrm{Aut}(T)_{\omega,\omega'}$	$\mathrm{Aut}(T)_\omega$	H	$\mathrm{Aut}(T)$

Figure 4. Propotypical examples of groups acting on trees

As a mnemonic for these types, it is helpful to draw the 3-regular tree in a way that suggests the action type, see Figure 5. The term *horocyclic* stems from the fact that such groups preserve the *horocycles*, sets of vertices that are equidistant (relative to a base vertex) from the chosen end, indicated by dashed lines in Figure 5.

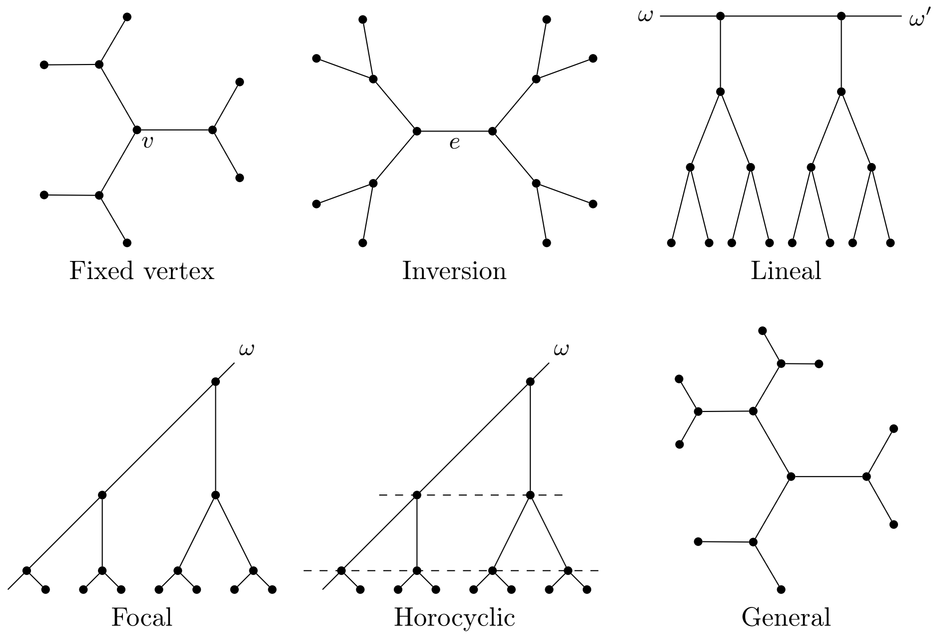

FIGURE 5. Drawing the 3-regular tree to suggest the action type.

Note that when T is locally finite, groups of type (*Fixed Vertex*) and (*Inversion*) are necessarily compact in the permutation topology as they act with finite vertex orbits. Vertex-transitive focal groups are also known as *scale groups*, see [Wil20]. They correspond both to self-replicating groups acting on rooted trees and elements of t.d.l.c. groups with non-trivial scale.

With the goal of classifying (subclasses of) groups acting on trees in some form, additional properties are typically assumed. This includes both independence and transitivity properties, which we elaborate on below.

2.1. **Independence Properties.** Tits [Tit70] first introduced an independence property named Property (P) for groups acting on trees in order to prove the simplicity of certain such groups.

This property was later generalised to Property (P_k), where $k \in \mathbb{N}_0$, by Banks–Elder–Willis [BEW15]. For closed subgroups, Tits' Property (P) coincides with Property (P_1) and the original simplicity criterion is generalised.

In hindsight, the Properties (P_k) also serve as an organising principle for groups acting on trees, which we elaborate on below. There are two essentially equivalent descriptions of these properties. The easier-to-state version reads as follows.

Definition 2.3. Let T be a tree, $H \leq \mathrm{Aut}(T)$, and $k \in \mathbb{N}_0$. The (P_k)-closure of H is

$$H^{(P_k)} := \{g \in \mathrm{Aut}(T) \mid \forall v \in VT\ \exists h \in H:\ g|_{B(v,k)} = h|_{B(v,k)}\}.$$

The group H has **Property** (P_k), or is (P_k)**-closed**, if $H = H^{(P_k)}$.

Note that the (P_0)-closure of $H \leq \mathrm{Aut}(T)$ is the largest subgroup of $\mathrm{Aut}(T)$ with the same vertex orbits as H. Section 3 introduces a large family of examples of groups with Property (P_k) for a given $k \in \mathbb{N}_0$. For now, consider the following.

Example 2.4.

(i) The group $\mathrm{Aut}(T_d)$ satisfies Property (P_0). Moreover, whenever $H \leq \mathrm{Aut}(T_d)$ is vertex-transitive, then $H^{(P_0)} = \mathrm{Aut}(T_d)$.

(ii) The group $\mathrm{Aut}(T_d)^+$ is the largest subgroup of $\mathrm{Aut}(T_d)$ which preserves the natural bipartition of VT_d and therefore satisfies Property (P_0).

(iii) The examples of Figure 4 satisfy Property (P_1). We have $\mathrm{Aut}(T)_\omega^{(P_0)} = \mathrm{Aut}(T)$ and $\mathrm{Aut}(T)_{\omega,\omega'}^{(P_0)} = \mathrm{Aut}(T)_{\{\omega,\omega'\}}$. The other examples are even (P_0)-closed.

(iv) Let $l : ET_d \to \Omega$ be a regular labelling of T_d, that is, for every vertex $v \in VT_d$ the map $l_v : \{e \in ET_d \mid o(e) = v\} \to \Omega$ induced by l is a bijection, and $l(e) = l(\overline{e})$ for all $e \in ET_d$. The *local action* of $g \in \mathrm{Aut}(T_d)$ at a vertex $v \in VT_d$ is the permutation $\sigma_1(g, v) := l_{gv} \circ g \circ l_v^{-1} \in \mathrm{Sym}(\Omega)$. Now consider the group of automorphisms whose local action is constant:

$$D := \{g \in \mathrm{Aut}(T_d) \mid \exists \tau \in \mathrm{Sym}(\Omega) : \ \forall v \in VT_d : \ \sigma_1(g, v) = \tau\}.$$

Then $D^{(P_0)} = D^{(P_1)} = \mathrm{Aut}(T_d)$ but $D^{(P_2)} = D$, so D satisfies Property (P_2).

(v) The group $\mathrm{PGL}(2, \mathbb{Q}_p)$ acting on its Bruhat-Tits tree T_{p+1} does not satisfy Property (P_k) for any $k \in \mathbb{N}_0$, see [BEW15, Section 4.1].

Among other properties, the (P_k)-closures of a given group naturally form a descending chain of overgroups converging to the group's closure.

Proposition 2.5. Let T be a tree, $H \leq \mathrm{Aut}(T)$, and $k, l \in \mathbb{N}_0$. Then

(i) $H^{(P_k)} \leq \mathrm{Aut}(T)$ is closed,

(ii) $H^{(P_0)} \geq H^{(P_1)} \geq H^{(P_2)} \geq \cdots \geq H^{(P_k)} \geq \cdots \geq \overline{H} \geq H$,

(iii) $\bigcap_{k\in\mathbb{N}_0} H^{(P_k)} = \overline{H}$, and

(iv) $\left(H^{(P_k)}\right)^{(P_l)} = H^{(P_l)}$, whenever $l \leq k$. In particular, $H^{(P_k)}$ is (P_k)-closed.

Proof. To see that $H^{(P_k)} \leq \mathrm{Aut}(T)$ is closed, let $g \in \mathrm{Aut}(T)\backslash H^{(P_k)}$. Then there is some vertex $v \in VT$ such that g does not coincide with any element of H on $B(v, k)$. Then the set $g\,\mathrm{Aut}(T)_{B(v,k)}$ of automorphisms of T that coincide with g on $B(v, k)$ is an open set containing g and contained in the complement of H.

The inclusions $H^{(P_0)} \geq H^{(P_1)} \geq H^{(P_2)} \geq \cdots \geq H^{(P_k)} \geq \cdots \geq H$ are immediate from the definition. The statements about $\overline{H}$ follow as $H^{(P_k)}$ is closed by part (i).

By part (ii), we have $\bigcap_{k\in\mathbb{N}_0} H^{(P_k)} \supseteq \overline{H}$. For the converse inclusion, suppose $g \in \bigcap_{k\in\mathbb{N}_0} H^{(P_k)}$. By definition, any of the basic open neighbourhoods $g\,\mathrm{Aut}(T)_{B(v,k)}$ $(k \in \mathbb{N})$ of g in $\mathrm{Aut}(T)$ contains an element $h \in H$. Hence the assertion.

Finally, we have $H \leq H^{(P_k)}$ and therefore $H^{(P_l)} \leq \left(H^{(P_k)}\right)^{(P_l)}$. The converse inclusion follows from the definition as $B(v, l) \subseteq B(v, k)$ for all $v \in VT$. □

By Proposition 2.5, a closed group $H \leq \mathrm{Aut}(T)$ can be recovered from the sequence of its (P_k)-closures through intersection. Since (P_k)-closures are always (P_k)-closed, a complete parametrisation of all (P_k)-closed groups would therefore entail a description of all closed subgroups of $\mathrm{Aut}(T)$. See Figure 6 for an illustration of this in the case of vertex-transitive subgroups of $\mathrm{Aut}(T_3)$. In milestone work, Reid–Smith [RS20] parametrised all (P_1)-closed groups, subsuming the Burger–Mozes [BM00, Section 3.2] universal groups $\mathrm{U}_1(F)$ and his own [Smi17] bi-universal groups. Moreover, Tornier [Tor23] described a subclass of all (P_k)-closed groups, known as generalised universal groups $\mathrm{U}_k(F)$, see Section 3. A more ambitious description of all (P_k)-closed groups is being pursued by various researchers.

The second, essentially equivalent version of Property (P_k) resembles the original definition of Tits more closely, and better motivates the term *independence*.

Let $e = \{v, w\}$ be an edge of T and $k \in \mathbb{N}$. We let $e^k := B(v, k) \cap B(w, k)$ denote the $(k-1)$-neighbourhood of e. Furthermore, let T_v and T_w denote the half-trees defined by e containing v and w respectively, see Figure 7.

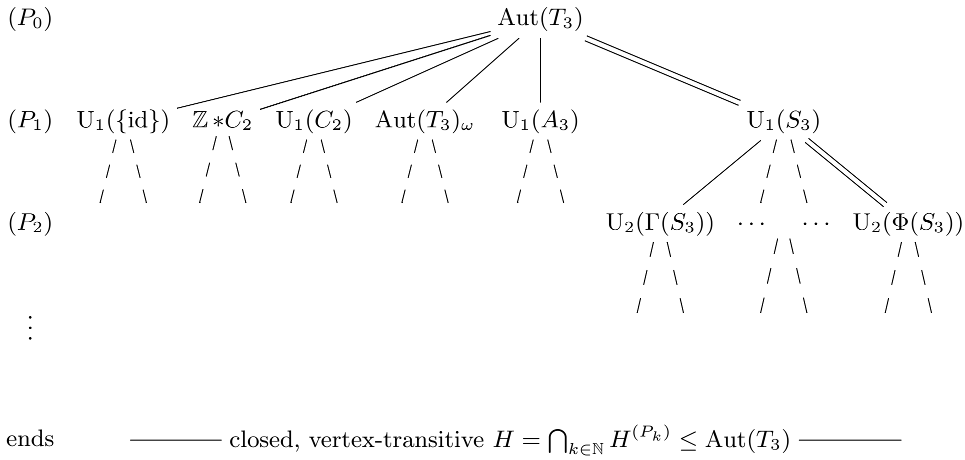

FIGURE 6. Recovering closed groups acting on trees from (P_k)-closures.

Definition 2.6. Let T be a tree, $H \leq \mathrm{Aut}(T)$, and $k \in \mathbb{N}$. The group H satisfies **Property (IP_k)** if for every edge $e = \{v, w\} \in ET$ we have

$$H_{e^k} = H_{e^k, T_w} \cdot H_{e^k, T_v}.$$

In words, the stabiliser of e^k in H acts independently on the two half-trees defined by the edge e when H has Property (IP_k). It is related to Property (P_k) as follows.

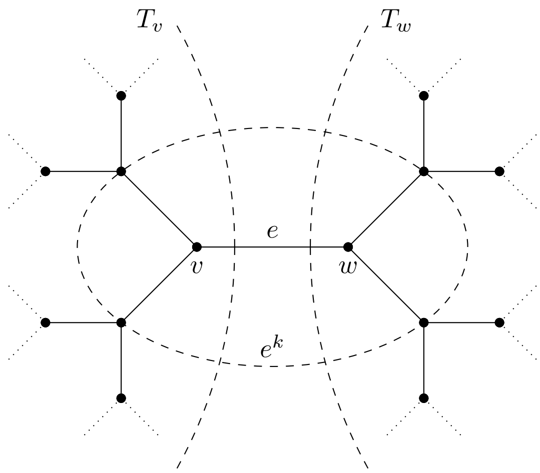

FIGURE 7. Illustration of Property (IP_k) in the case of $k = 2$.

Theorem 2.7. Let T be a tree, $H \leq \mathrm{Aut}(T)$ and $k \in \mathbb{N}$.

(i) If H satisfies Property (IP_k) then $H^{(P_k)} = \overline{H}$.
(ii) If $H^{(P_k)} = \overline{H}$ then $\overline{H}$ satisfies Property (IP_k).

In particular, for closed $H \leq \mathrm{Aut}(T)$, Property (P_k) and (IP_k) are equivalent.

Proof. First, suppose that H satisfies Property (IP_k). We prove by induction on $r \geq k$ that $H^{(P_r)} = H^{(P_k)}$. The induction base is immediate. For the induction step, note that $H^{(P_r)} \leq H^{(P_k)}$ by definition. Now, let $g \in H^{(P_k)}$ and $v \in VT$. We construct element $h \in H$ such that $g|_{B(v,r+1)} = h|_{B(v,r+1)}$. That is, $g \in H^{(P_{r+1})}$.

By the induction hypothesis, $g \in H^{(P_r)}$ and hence there is $h' \in H$ such that $g|_{B(v,r)} = h'|_{B(v,r)}$. Therefore, $h'^{-1}g \in H^{(P_r)} = H^{(P_k)}$ fixes $B(v,r)$. Let $v_1, \dots, v_m$ be the vertices at distance $r-k+1$ from v. For all $i \in \{1,\dots,m\}$ there is $a_i \in H$ such that $a_i|_{B(v_i,k)} = (h'^{-1}g)|_{B(v_i,k)}$. In particular, a_i fixes the set $B(v_i,k) \cap B(v,r)$. Let $w_i \in VT$ denote the vertex adjacent to v_i that is closest to v. Then we have $B(v_i,k) \cap B(v,r) = B(v_i,k) \cap B(w_i,k) = \{v_i,w_i\}^k$. Since H satisfies Property (IP_k), we may write $a_i = b_i c_i$ for some $b_i, c_i \in H$ fixing $\{v_i,w_i\}^k$ and acting only on the half-tree including v_i and w_i respectively. Now, the element $b_1 \cdots b_m$ fixes $B(v,r)$ and acts like $h'^{-1}g$ on $B(v_i,r)$ for all $i \in \{1,\dots,m\}$. Therefore, $(h'b_1 \cdots b_m)^{-1}g \in H$ fixes $B(v,r+1)$ and we may thus put $h := h'b_1 \cdots b_m$.

Conversely, suppose that $H^{(P_k)} = \overline{H}$. We show that $H^{(P_k)}$ satisfies Property (IP_k). Let $e = \{v,w\} \in ET$. By definition, $H^{(P_k)}_{e^k} \supseteq H^{(P_k)}_{e^k,T_v} \cdot H^{(P_k)}_{e^k,T_w}$. Conversely, let $h \in H^{(P_k)}_{e^k}$. Define $h_v \in H^{(P_k)}_{e^k,T_w}$ by setting $h_v|_{B(x,k)} = \mathrm{id}$ for all $x \in VT_w$ and $h_v|_{B(x,k)} = h$ for all $x \in VT_v$. Similarly, define $h_w \in H^{(P_k)}_{e^k,T_v}$ to only act on T_w. Then $h = h_v h_w$ and both h_v and h_w are elements of $H^{(P_k)}$. □

2.2. **Transitivity Properties.** This section contains a brief survey of classification results concerning groups acting on trees with several transitivity properties.

2.2.1. *Local transitivity.* Transitivity assumptions of local nature are among the most prevalent and allow for (finite) permutation group theory methods to be used.

Definition 2.8. Let T be a tree, $G \leq \mathrm{Aut}(T)$ and (X) a property of permutation groups. The group G is **locally (X)** if for every vertex $v \in VT$ the permutation group induced by G_v on $o^{-1}(v)$ satisfies (X).

For example, property (X) in Definition 2.8 may represent (semi)regularity, transitivity, (semi)primitivity, 2-transitivity, or admitting a finite base.

- Burger–Mozes [BM00] characterised the locally transitive, (P_1)-closed subgroups of $\mathrm{Aut}(T_d)$ $(d \in \mathbb{N}_{\geq 3})$ that contain an edge inversion as universal groups $\mathrm{U}_1(F)$.
- The author [Tor23] characterised the locally transitive, (P_k)-closed subgroups $(k \in \mathbb{N})$ of $\mathrm{Aut}(T_d)$ $(d \in \mathbb{N}_{\geq 3})$ that contain an *involutive* edge inversion as generalised universal groups, see Section 3.
- Smith [Smi17] characterised the locally transitive, (P_1)-closed subgroups of $\mathrm{Aut}(T_{m,n})$ $(m,n \in \mathbb{N}_{\geq 2})$ preserving the bipartition as universal groups $\mathrm{U}(F_1,F_2)$.

2.2.2. *Boundary Transitivity.* Groups acting on trees that act transitively on the boundary are much studied as well.

- Radu [Rad17] characterised boundary-2-transitive subgroups of $\mathrm{Aut}(T_{m,n})$, for $m,n \in \mathbb{N}_{\geq 6}$, that locally contain the alternating group through various families.
- Reid [Rei23] has laid ground work towards generalising Radu's classification to 2-transitive local actions other than the alternating group.
- Semal [Sem24] classified the irreducible unitary representations of the groups studied by Radu [Rad17].

2.2.3. *Global transitivity.* Figure 6 lays out a strategy to classify vertex-transitive groups acting on trees. Groups acting transitively on edges, or more generally, paths of length $s \in \mathbb{N}$, so called s-arc-transitive groups, play a crucial role in the context of discrete groups and the Weiss conjecture [Wei78].

2.2.4. *No transitivity.* In milestone work, generalising some of the classification results for locally transitive groups mentioned in Section 2.2.1, Reid–Smith [RS20] recently parametrised all (P_1)-closed groups acting on (not necessarily locally finite) trees using graph-based combinatorial structures known as *local action diagrams*. It is several researchers' ambition to achieve the same for (P_k)-closed groups.

3. Generalised Universal Groups

In this section, we introduce a versatile class of groups acting on the regular tree T_d ($d \in \mathbb{N}_{\geq 3}$) that satisfy Property (P_k) for a given $k \in \mathbb{N}$.

Let Ω be a set of cardinality $d \in \mathbb{N}_{\geq 3}$ and let T_d denote the d-regular tree. A **regular labelling** l of T_d is a map $l : ET_d \to \Omega$ such that for every $v \in VT_d$ the map $l_v : o^{-1}(v) \to \Omega,\ e \mapsto l(e)$ is a bijection, and $l(e) = l(\overline{e})$ for all $e \in E$.

For every $k \in \mathbb{N}$, fix a tree $B_{d,k}$ that is isomorphic to a ball of radius k around a vertex in T_d. Let b denote its center and carry over the labelling of T_d to $B_{d,k}$ via the chosen isomorphism. Then for every $v \in V$ there is a unique, label-respecting isomorphism $l_v^k : B(v,k) \to B_{d,k}$. We define the k**-local action** $\sigma_k(g,v) \in \mathrm{Aut}(B_{d,k})$ of an automorphism $g \in \mathrm{Aut}(T_d)$ at a vertex $v \in V$ via

$$\sigma_k : \mathrm{Aut}(T_d) \times VT_d \to \mathrm{Aut}(B_{d,k}),\ (g,v) \mapsto \sigma_k(g,x) := l_{gv}^k \circ g \circ (l_v^k)^{-1}.$$

Definition 3.1. Let $F \leq \mathrm{Aut}(B_{d,k})$ and l be a labelling of T_d. Define

$$\mathrm{U}_k^{(l)}(F) := \{g \in \mathrm{Aut}(T_d) \mid\ \forall x \in VT_d :\ \sigma_k(g,x) \in F\}.$$

In the case of $k = 1$, Definition 3.1 is due to Burger–Mozes [BM00, Section 3]. For $k \geq 2$, it stems from [Tor23, Definition 4.1].

Figure 8 illustrates the definition of the maps σ_k, resembling transition maps in differential geometry, and thereby the definition of the groups $\mathrm{U}_k(F)$.

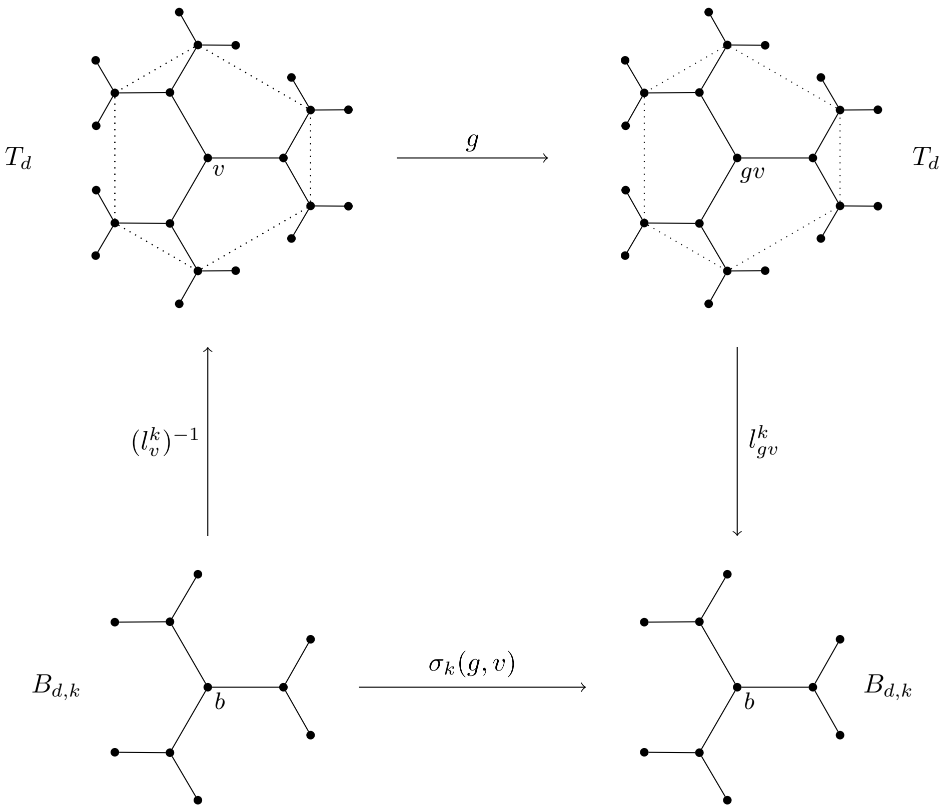

Figure 8. Illustration of the definition of $\mathrm{U}_k(F)$ for $(d,k) = (3,2)$.

One can show that for different labellings of the tree T_d, the corresponding universal groups are conjugate in $\mathrm{Aut}(T_d)$. We therefore often omit the reference to an explicit labelling. The group $\mathrm{U}_k(F)$ satisfies the following basic properties.

Proposition 3.2. Let $F \leq \mathrm{Aut}(B_{d,k})$. The group $\mathrm{U}_k(F)$ is

(i) closed in $\mathrm{Aut}(T_d)$,
(ii) vertex-transitive,
(iii) compactly generated, and
(iv) satisfies Property (IP_k) and (P_k).

Proof. As to (i), note that if $g \notin \mathrm{U}_k(F)$ then $\sigma_k(g,v) \notin F$ for some $v \in VT_d$. In this case, the open neighbourhood $\{h \in \mathrm{Aut}(T_d) \mid h|_{B(v,k)} = g|_{B(v,k)}\}$ of g in $\mathrm{Aut}(T_d)$ is also contained in the complement of $\mathrm{U}_k(F)$.

For (ii), let $v, v' \in V$ and let $g \in \mathrm{Aut}(T_d)$ be the colour-preserving automorphism of T_d, with $gv = v'$. Then $g \in \mathrm{U}_k(F)$ as $\sigma_k(g,v) = \mathrm{id} \in F$ for all $v \in V$.

The proof of part (iii) follows the same argument as Proposition 1.5.

For part (iv), it suffices to show either property due to Theorem 2.7 and part (i). It is immediate from the definition that $\mathrm{U}_k(F)$ satisfies Property (P_k). □

3.1. **Compatibility.** While the definition of $\mathrm{U}_k(F)$ allows for k-local actions to be in $F \leq \mathrm{Aut}(B_{d,k})$, compatibility issues between neighbouring vertices may prevent some elements of F from occuring as a k-local action of an element of $\mathrm{U}_k(F)$.

To make this precise, we say that $\mathrm{U}_k(F)$ **k-locally acts** like F if the actions $\mathrm{U}_k(F)_v \curvearrowright B(v,k)$ and $F \curvearrowright B_{d,k}$ are isomorphic for every $v \in V$ via the label-respecting isomorphism l_v^k. Whereas this holds for $k=1$ by [BM00, Section 3.2], it need not be true for $k \geq 2$. In the following, given a vertex v in a tree labelled by Ω and a label $\omega \in \Omega$, we let v_ω denote the neighbour of v along the edge labelled ω. Now, given $F \leq \mathrm{Aut}(B_{d,k})$, an element $\alpha \in F$ and $\omega \in \Omega$, put

$$C_F(\alpha,\omega) := \{\alpha_\omega \in F \mid \sigma_{k-1}(\alpha_\omega, b) = \sigma_{k-1}(\alpha, b_\omega) \text{ and } \sigma_{k-1}(\alpha_\omega, b_\omega) = \sigma_{k-1}(\alpha, b)\},$$

the set of all elements in F that are **compatible with α in direction ω**.

Proposition 3.3. Let $d \in \mathbb{N}_{\geq 3}$, $k \in \mathbb{N}$ and $F \leq \mathrm{Aut}(B_{d,k})$. The group $\mathrm{U}_k(F)$ k-locally acts like F if and only if F satisfies

$$\text{(C)} \qquad \forall \alpha \in F \ \forall \omega \in \Omega : \ C_F(\alpha,\omega) \neq \varnothing.$$

Proof. First, suppose that the group $\mathrm{U}_k(F)$ does k-locally act like F. Let $v \in VT_d$. Then for every $\alpha \in F$ there is an element $g \in \mathrm{U}_k(F)_v$ that satisfies $\sigma_k(g,v) = \alpha$. Now, let $\omega \in \Omega$ and consider the neighbour v_ω of v. Note that

$$B(v,k) \cap B(v_\omega,k) = B(v,k-1) \cup B(v_\omega,k-1).$$

The restrictions of g to $B(v,k)$ and $B(v_\omega,k)$ necessarily agree on the intersection $B(v,k) \cap B(v_\omega,k)$, see Figure 9. Since we have $B(v_\omega,k-1) = (l_v^k)^{-1}(B(b_\omega,k-1))$ and $B(v,k-1) = (l_{v_\omega}^k)^{-1}(B(b_\omega,k-1))$ we conclude that $\sigma_k(g,v_\omega) \in C_F(\alpha,\omega)$.

Conversely, suppose that $C_F(\alpha,\omega)$ is non-empty for all $\alpha \in F$ and $\omega \in \Omega$. Let $v, w \in VT_d$. An element $g \in \mathrm{U}_k(F)$ that satisfies $\sigma_k(g,v) = \alpha$ and $gv = w$ is readily constructed: define $g : B(v,k) \to B(w,k)$ by setting $gv := w$ and $\sigma_k(g,v) := \alpha$. Then, given elements $\alpha_\omega \in C_F(\alpha,\omega)$ for all $\omega \in \Omega$, there is a unique extension of g to $B(v,k+1)$ such that $\sigma_k(g,v_\omega) = \alpha_\omega$ for all $\omega \in \Omega$. Proceed inductively. □

The following example illlustrates the problem when F does not satisfy (C).

Example 3.4. Let $\Omega := \{1,2,3\}$ and $k := 2$. Define $\alpha \in \mathrm{Aut}(B_{3,2})$ to be the element that swaps the leaves b_{12} and b_{13} of $B_{3,2}$. Put $F := \langle \alpha \rangle = \{\mathrm{id}, \alpha\}$. Then $C_F(\alpha,1) = \varnothing$ and hence F does not satisfy (C).

However, one can show that it suffices to check (C) on generators, which results in the following, see [Tor23, Proposition 4.10]. In Example 3.4, we have $C(F) = \{\mathrm{id}\}$.

Proposition 3.5. Let $F \leq \mathrm{Aut}(B_{d,k})$. Then F has a unique maximal subgroup $C(F)$ that satisfies (C). We have $C(C(F)) = C(F)$ and $\mathrm{U}_k(F) = \mathrm{U}_k(C(F))$.

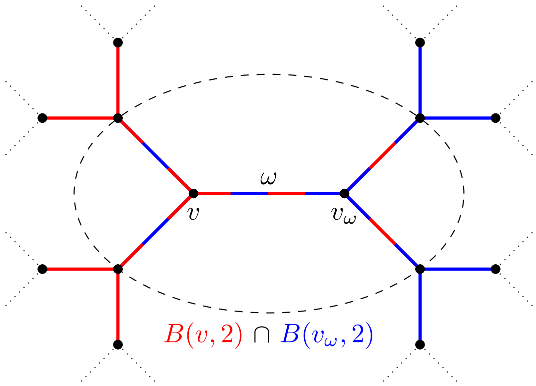

FIGURE 9. Two adjacent k-balls intersect in the union of two $(k-1)$-balls.

3.2. **Examples.** For clarity, we only consider the case $k=2$. In certain situations, this is actually sufficient, see [Tor23, Theorem 4.32]. In view of the compatibility between elements of $\mathrm{Aut}(B_{d,k})$ depending on their $(k-1)$-local actions, see Figure 9, we identify $\mathrm{Aut}(B_{d,k})$ with its image under the map

$$\mathrm{Aut}(B_{d,k}) \to \mathrm{Aut}(B_{d,k-1}) \ltimes \prod_{\omega\in\Omega} \mathrm{Aut}(B_{d,k-1}),\ \alpha \mapsto (\sigma_{k-1}(\alpha,b),(\sigma_{k-1}(\alpha,b_\omega))_\omega),$$

where $\mathrm{Aut}(B_{d,k-1})$ acts on $\prod_{\omega\in\Omega}\mathrm{Aut}(B_{d,k-1})$ by permuting the factors according to its action on $S(b,1)\cong\Omega$. That is, multiplication in $\mathrm{Aut}(B_{d,k})$ is given by

$$(\alpha,(\alpha_\omega)_{\omega\in\Omega})\circ(\beta,(\beta_\omega)_{\omega\in\Omega}) = (\alpha\beta,(\alpha_{\beta\omega}\beta_\omega)_{\omega\in\Omega}).$$

In particular, we have

$$\mathrm{Aut}(B_{d,2}) = \{(a,(a_\omega)_{\omega\in\Omega}) \mid a\in\mathrm{Sym}(\Omega),\ \forall\omega\in\Omega:\ a_\omega\in\mathrm{Sym}(\Omega) \text{ and } a_\omega\omega = a\omega\}.$$

To define examples of local actions $F\le\mathrm{Aut}(B_{d,2}$ that satisfy (C), consider the map $\gamma:\mathrm{Sym}(\Omega)\to\mathrm{Aut}(B_{d,2})$, $a\mapsto(a,(a,\dots,a))$. Given $F\le\mathrm{Sym}(\Omega)$, the group

$$\Gamma(F) := \mathrm{im}(\gamma|_F) = \{(a,(a,\dots,a)) \mid a\in F\} \cong F$$

is a subgroup of $\mathrm{Aut}(B_{d,2})$ which is isomorphic to F and satisfies (C): all elements of $\Gamma(F)$ are self-compatible in every direction, i.e., $\gamma(a)\in C_{\Gamma(F)}(\gamma(a),\omega)$ for all $a\in F$ and $\omega\in\Omega$. Note that $\mathrm{U}_2(\Gamma(F))=\{\alpha\in\mathrm{Aut}(T_d)\mid \exists a\in F:\forall x\in V:\ \sigma_1(\alpha,x)=a\}$, generalising Example 2.4 (iv). Next, consider the group

$$\Phi(F) := \{(a,(a_\omega)_\omega) \mid a\in F,\ \forall\omega\in\Omega:\ a_\omega\in C_F(a,\omega)\} \cong F\ltimes\prod_{\omega\in\Omega}F_\omega.$$

It naturally satisfies condition (C) and $\mathrm{U}_2(\Phi(F))=\mathrm{U}_1(F)$ for every $F\le\mathrm{Sym}(\Omega)$.

The following kind of 2-local action is related to the sign construction in [Rad17]. Let $F\le\mathrm{Sym}(\Omega)$ and $\rho:F\twoheadrightarrow A$ be a homomorphism to an abelian group A. Define

$$\Pi(F,\rho,\{1\}) := \left\{(a,(a_\omega)_\omega)\in\Phi(F) \,\middle|\, \prod_{\omega\in\Omega}\rho(a_\omega)=1\right\},\ \text{and}$$
$$\Pi(F,\rho,\{0,1\}) := \left\{(a,(a_\omega)_\omega)\in\Phi(F) \,\middle|\, \rho(a)\prod_{\omega\in\Omega}\rho(a_\omega)=1\right\}.$$

Proposition 3.6. Let $F\le\mathrm{Sym}(\Omega)$ and let $\rho:F\twoheadrightarrow A$ be a homomorphism to an abelian group A. Further, let $\widetilde{F}\in\{\Pi(F,\rho,\{1\}),\Pi(F,\rho,\{0,1\})\}$. If $\rho(F_\omega)=A$ for all $\omega\in\Omega$ then $\pi\widetilde{F}=F$ and $\widetilde{F}$ satisfies (C).

Proof. As $C_F(a,\omega)=aF_\omega$, and $\rho(F_\omega)=A$ for all $\omega\in\Omega$, an element $(a,(a_\omega)_\omega)\in\Phi(F)$ can be turned into an element of $\widetilde{F}$ by changing a_ω for a single, arbitrary $\omega\in\Omega$. We conclude that $\pi\widetilde{F}=F$ and that $\widetilde{F}$ satisfies (C). $\square$

There are many more families of examples of local actions that satisfy (C), both in the case $k=2$ and for larger values of k. Figure 10 shows what is known about locally transitive ones up to $k=4$ in the case of T_3. There is a comprehensive GAP package which implements these families of examples, see [HT23] and [FTW25].

3.3. **Universality.** The groups $\mathrm{U}_k(F)$ are universal in the sense of the following maximality statement, which should be compared to [BM00, Proposition 3.2.2].

Theorem 3.7. Let $H \leq \mathrm{Aut}(T_d)$ be locally transitive and contain an involutive inversion. Then there is a labelling l of T_d such that

$$\mathrm{U}_1^{(l)}(F^{(1)}) \geq \mathrm{U}_2^{(l)}(F^{(2)}) \geq \cdots \geq \mathrm{U}_k^{(l)}(F^{(k)}) \geq \cdots \geq H \geq \mathrm{U}_1^{(l)}(\{\mathrm{id}\})$$

where $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ is action isomorphic to the k-local action of H.

Proof. First, we construct a labelling l of T_d such that $H \geq \mathrm{U}_1^{(l)}(\{\mathrm{id}\})$: Fix $x \in V$ and choose a bijection $l_x : E(x) \to \Omega$. By the assumptions, there is an involutive inversion $\iota_\omega \in H$ of the edge $(x, x_\omega) \in E$ for every $\omega \in \Omega$. Using these inversions, we define the announced labelling inductively: Set $l|_{E(x)} := l_x$ and assume that l is defined on $E(x,n)$. For $e \in E(x,n+1)\backslash E(x,n)$ put $l(e) := l(\iota_\omega(e))$ if x_ω is part of the unique reduced path from x to $o(e)$. Since the ι_ω $(\omega \in \Omega)$ have order 2, we obtain $\sigma_1(\iota_\omega, y) = \mathrm{id}$ for all $\omega \in \Omega$ and $y \in V$. Therefore, $\langle \{\iota_\omega \mid \omega \in \Omega\}\rangle = \mathrm{U}_1^{(l)}(\{\mathrm{id}\}) \leq H$.

Now, let $h \in H$ and $y \in V$. Further, let $(x, x_1, \ldots, x_n, y)$ and $(x, x_1', \ldots, x_m', h(y))$ be the unique reduced paths from x to y and $h(y)$ respectively. Since $\mathrm{U}_1^{(l)}(\{\mathrm{id}\}) \leq H$, the group H contains the label-respecting inversion ι_e of every edge $e \in E$. Hence

$$s := \iota_{(x_1',x)}^{-1} \cdots \iota_{(x_m',x_{m-1}')}^{-1} \iota_{(h(y),x_m')}^{-1} \circ h \circ \iota_{(y,x_n)} \cdots \iota_{(x_2,x_1)} \iota_{(x_1,x)} \in H.$$

Also, s stabilizes x. The cocycle identity implies for every $k \in \mathbb{N}$:

$$\sigma_k(h,y) = \sigma_k(\iota_{(h(y),x_m')} \cdots \iota_{(x_1',x)} \circ s \circ \iota_{(x_1,x)}^{-1} \cdots \iota_{(y,x_n)}^{-1}, y) = \sigma_k(s,x) \in F^{(k)}.$$

where $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ is defined by $l_x^k \circ H_x|_{B(x,k)} \circ (l_x^k)^{-1}$. □

The universality statement helps to characterise locally transitive generalised universal groups precisely as those locally transitive subgroups of $\mathrm{Aut}(T_d)$ that contain an edge inversion of order 2 and satisfy Property (P_k) for some $k \in \mathbb{N}$.

Theorem 3.8. Let $H \leq \mathrm{Aut}(T_d)$ be locally transitive and contain an involutive inversion. Then $H^{(P_k)} = \mathrm{U}_k^{(l)}(F^{(k)})$ for some labelling l of T_d and $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$.

Proof. Let l and $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ be as in Theorem 3.7. Then $H^{(P_k)} = \mathrm{U}_k^{(l)}(F^{(k)})$:

Let $g \in \mathrm{U}_k(F^{(k)})$ and $x \in V$. Since $\mathrm{U}_1^{(l)}(\{\mathrm{id}\}) \leq H$ there is $h' \in \mathrm{U}_1^{(l)}(\{\mathrm{id}\}) \leq H$ with $h'(x) = g(x)$, and since H is k-locally action isomorphic to $F^{(k)}$ there is $h'' \in H_x$ such that $\sigma_k(h'',x) = \sigma_k(g,x)$. Then $h := h'h'' \in H$ satisfies $g|_{B(x,k)} = h|_{B(x,k)}$.

Conversely, let $g \in H^{(P_k)}$. Then all k-local actions of g stem from elements of H. Given that $H \leq \mathrm{U}_k(F^{(k)})$ by Theorem 3.7, we conclude that $g \in \mathrm{U}_k(F^{(k)})$. □

Corollary 3.9. Let $H \leq \mathrm{Aut}(T_d)$ be closed, locally transitive and contain an involutive inversion. Then $H = \mathrm{U}_k^{(l)}(F^{(k)})$ for some labelling l of T_d and an action $F^{(k)} \leq \mathrm{Aut}(B_{d,k})$ if and only if H satisfies Property (P_k).

Proof. If $H = \mathrm{U}_k^{(l)}(F^{(k)})$ then H satisfies Property (P_k) by Proposition 3.2. Conversely, if H satisfies Property (P_k) then $H = \overline{H} = H^{(P_k)}$ by [BEW15, Theorem 5.4] and the assertion follows from Theorem 3.8. □

Example 3.10. The group $\mathrm{PGL}(2,\mathbb{Q}_p)$ acting on its Bruhat-Tits tree T_{p+1} does not satisfy Property (P_k) for any $k \in \mathbb{N}$. However, its (P_k)-closures are generalised universal groups. For example, it can be shown that

$$\mathrm{PGL}(2,\mathbb{Q}_p)^{(P_1)} = \mathrm{U}_1(\mathrm{PGL}(2,p)) \quad \text{and} \quad \mathrm{PGL}(2,\mathbb{Q}_2)^{(P_2)} = \mathrm{U}_2(\Phi(S_3)).$$

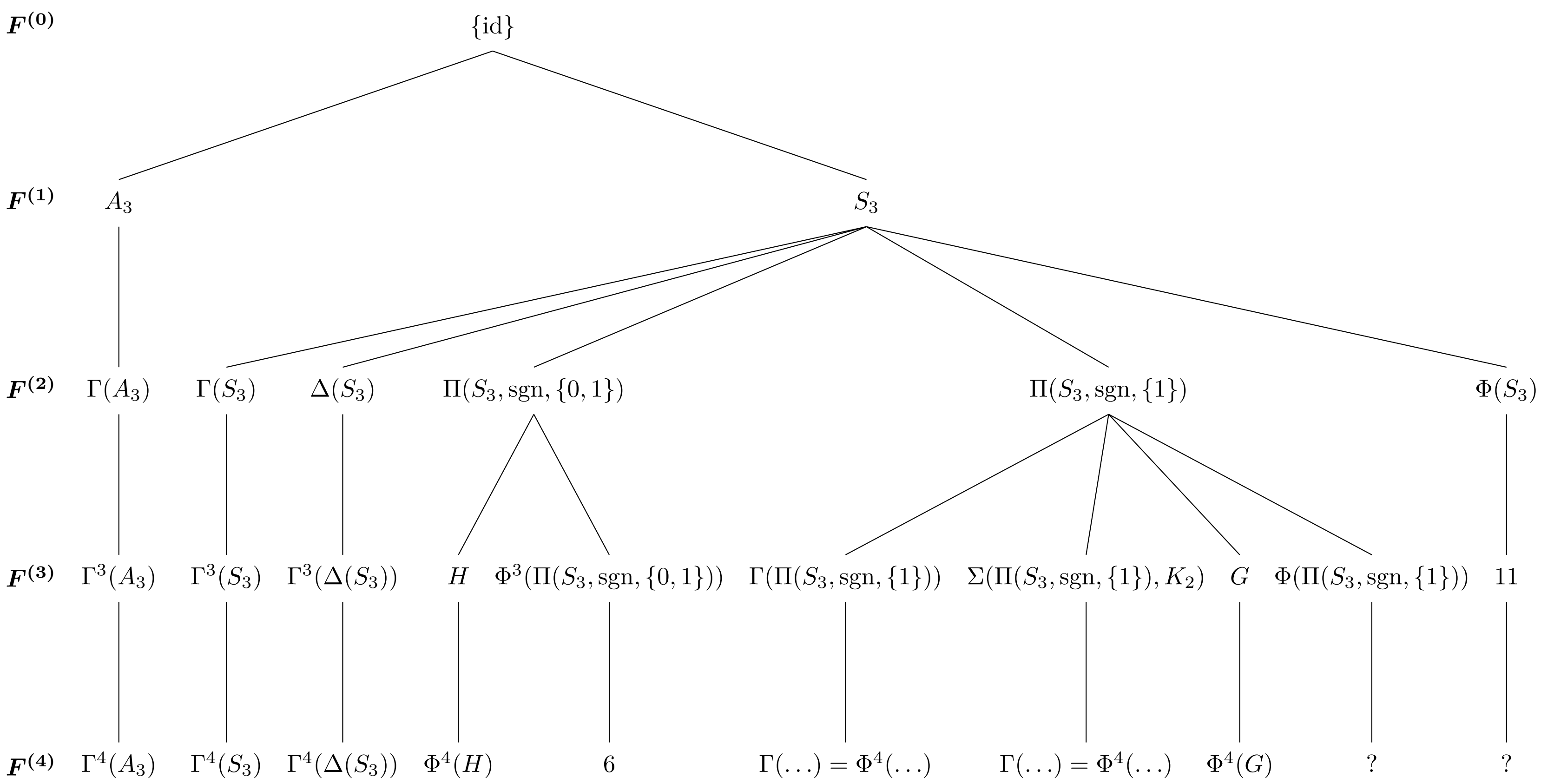

FIGURE 10. Conjugacy class representatives of subgroups of $\mathrm{Aut}(B_{3,k})$ with (C) for $k \in \{0,1,2,3,4\}$ and their relationship.

References

[Abe74] H. Abels, *Specker-kompaktifizierungen von lokal kompakten topologischen gruppen*, Mathematische Zeitschrift **135** (1974), 325–361.

[ÁLM23] A. Árnadóttir, W. Lederle, and R. Möller, *Cayley–abels graphs and invariants of totally disconnected, locally compact groups*, Journal of the Australian Mathematical Society **114** (2023), no. 2, 145–177.

[BEW15] C. Banks, M. Elder, and G. Willis, *Simple groups of automorphisms of trees determined by their actions on finite subtrees*, Journal of Group Theory **18** (2015), no. 2, 235–261.

[BM00] M. Burger and S. Mozes, *Groups acting on trees: from local to global structure*, Publications Mathématiques de l'IHÉS **92** (2000), no. 1, 113–150.

[FTW25] M. Ferov, S. Tornier, and G. Willis, *Computational aspects of totally disconnected locally compact groups*, Handbook of Visual, Experimental and Computational Mathematics: Bridges through Data, Springer, 2025, pp. 1–36.

[GGT18] A. Garrido, Y. Glasner, and S. Tornier, *Automorphism Groups of Trees: Generalities and Prescribed Local Actions*, New Directions in Locally Compact Groups (P.-E. Caprace and M. Monod, eds.), Cambridge University Press, 2018, pp. 92–116.

[Gle52] A. Gleason, *Groups without small subgroups*, Annals of Mathematics (1952), 193–212.

[HT23] K. Hannouch and S. Tornier, *UGALY, Universal Groups Acting LocallY, Version 4.1.3*, https://gap-packages.github.io/UGALY, July 2023, Refereed GAP package.

[KM08] B. Krön and R. Möller, *Analogues of Cayley graphs for topological groups*, Mathematische Zeitschrift **258** (2008), no. 3, 637.

[Led22] W. Lederle, *Cayley-abels graphs: Local and global perspectives*, arXiv preprint 2210.16284 (2022).

[MZ52] D. Montgomery and L. Zippin, *Small subgroups of finite-dimensional groups*, Proceedings of the National Academy of Sciences **38** (1952), no. 5, 440–442.

[Rad17] N. Radu, *A classification theorem for boundary 2-transitive automorphism groups of trees*, Inventiones mathematicae **209** (2017), no. 1, 1–60.

[Rei23] C. Reid, *Rigid stabilizers and local prosolubility for boundary-transitive actions on trees*, arXiv preprint 2301.09078 (2023).

[RS20] C. Reid and S. Smith, *Groups acting on trees with Tits' independence property (P)*, arXiv preprint 2002.11766 (2020).

[Sem24] L. Semal, *Radu groups acting on trees are ccr*, Journal of the Australian Mathematical Society **117** (2024), no. 2, 149–186.

[Smi17] S. Smith, *A product for permutation groups and topological groups*, Duke Mathematical Journal **166** (2017), no. 15, 2965–2999.

[Tit70] J. Tits, *Sur le groupe des automorphismes d'un arbre*, Essays on topology and related topics, Springer, 1970, pp. 188–211.

[Tor23] S. Tornier, *Groups acting on trees with prescribed local action*, Journal of the Australian Mathematical Society **115** (2023), no. 2, 240–288.

[Wei78] R. Weiss, *s-Transitive graphs*, Algebraic methods in graph theory **25** (1978), 827–847.

[Wes15] P. Wesolek, *Elementary totally disconnected locally compact groups*, Proceedings of the London Mathematical Society **110** (2015), no. 6, 1387–1434.

[Wes18] ———, *An introduction to totally disconnected locally compact groups*, Preprint (2018).

[Wil20] G. Willis, *Scale groups*, arXiv preprint 2008.05220 (2020).

[Yam53] H. Yamabe, *A generalization of a theorem of Gleason*, Annals of Mathematics (1953).