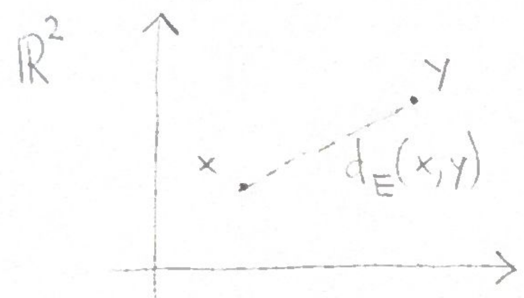
All distances are created equal

(BMath Meetup, $\approx$ 45 minutes, 02/03/26)

Measuring distance is a familiar concept:



"Euclidean distance"

Given an arbitrary set S, such as $\mathbb{R}^2$, what properties should a distance function d on S satisfy?

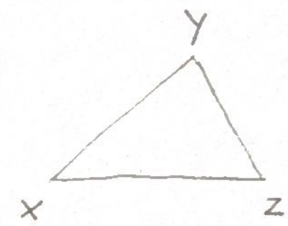
(i) $d(x,y) \in \mathbb{R}_{\geq 0}$

(ii) $d(x,y) = 0 \iff x = y$

(iii) $d(x,y) = d(y,x)$

(iv) $d(x,z) \leq d(x,y) + d(y,z)$

(for all $x,y,z \in S$)



Even on $S = \mathbb{R}^2$ this gives a lot of options:

(a) Given $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$ define

$$d_M((x_1,y_2),(x_2,y_2)) := |x_1 - x_2| + |y_1 - y_2| \in \mathbb{R}$$

Does this satisfy the distance axioms?

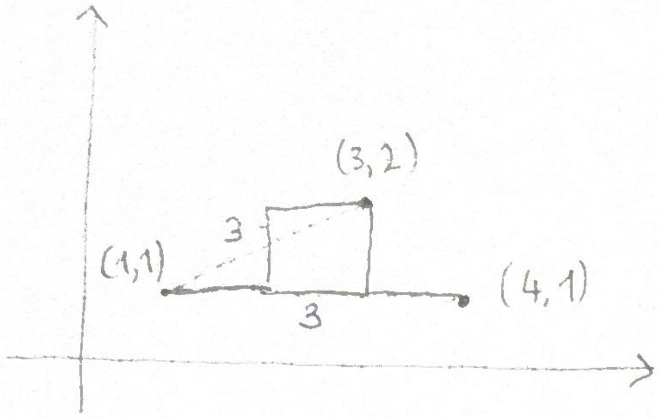
(i) ✓, (ii) ✓, (iii) ✓

(iv) Let's check:

$$d_M((x_1,y_1),(x_3,y_3)) \overset{?}{\leq} d_M((x_1,y_1),(x_2,y_2)) + d_M((x_2,y_2),(x_3,y_3))$$

$$\begin{array}{ccc} \| & & \| \\ |x_1 - x_3| + |y_1 - y_3| & & |x_1 - x_2| + |y_1 - y_2| + |x_2 - x_3| + |y_2 - y_3| \\ & & |x_1 - x_2| + |x_2 - x_3| + |y_1 - y_2| + |y_2 - y_3| \end{array}$$

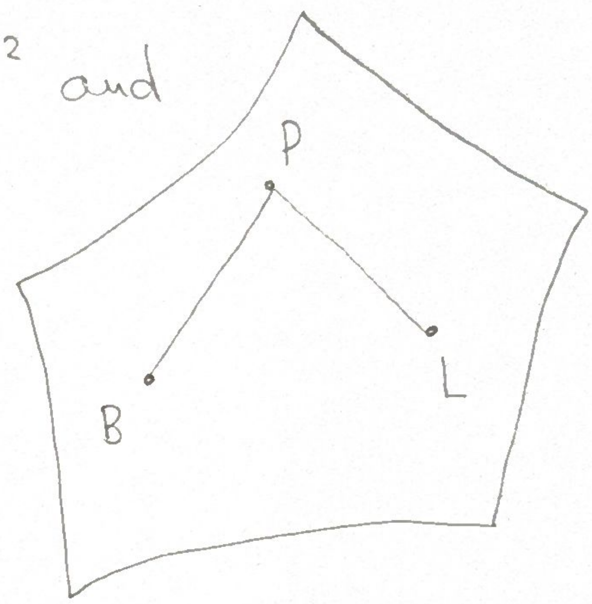
This works essentially because for $S = \mathbb{R}$, the assignment $d(x,y) = |x - y|$ is a distance function and hence satisfies (iv).



This is called the Manhattan metric.

Shortest paths go along a grid

(b) Again, let $S = \mathbb{R}^2$. Fix a point $P \in \mathbb{R}^2$ and define for all $B, L \in \mathbb{R}^2$:



$$d_F(B,L) := d_E(B,P) + d_E(P,L)$$

Does this satisfy the axioms?

(i) ✓, (ii) ✓, (iii) ✓

(iv) Let's check:

$$d_F(B,L) \overset{?}{\leq} d_F(B,Q) + d_F(Q,L)$$

$$\underline{d_E}(B,P) + \underline{d_E}(P,L) \qquad \underline{d_E(B,P)} + \underbrace{d_E(P,Q) + d_E(Q,P)}_{\geq 0} + \underline{d_E(P,L)}$$

What do the letters P, B, L stand for?

Hint 1: map. Hint 2: this is also called the SNCF-distance.

Hint 3: the F in d_F stands for "France".

(c) Let $S = \mathbb{Z}$ and choose a prime p.

For $x \in \mathbb{Z}$ define $m_p(x)$ to be the multiplicity of p in x.

(and $m_p(0) := \infty$)

Example: $p = 3$

$m_3(3) = 1$, $m_3(9) = 2$, $m_3(24) = 1$, $m_3(11) = 0$

$m_3(12) = 1$

Now define for $x, y \in \mathbb{Z}$:

$$d_p(x,y) := p^{-m_p(x-y)} \qquad (\text{or define } d_p(x,x) = 0)$$

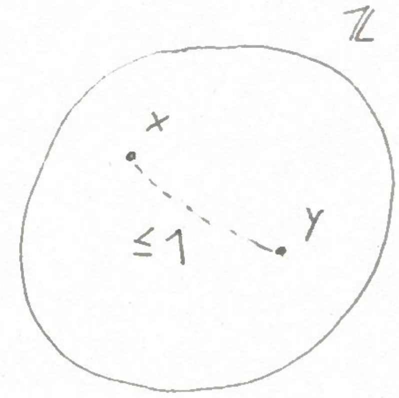
Example

$$d_3(3,12) = 3^{-m_3(-9)} = 3^{-2} = \frac{1}{9}$$

$$d_3(12,24) = 3^{-m_3(-12)} = 3^{-1} = \frac{1}{3}$$

$$d_3(24,11) = 3^{-m_3(13)} = 1$$

as far apart as possible!



Does this satisfy the axioms?

(i) ✓, (ii) ✓, (iii) $m_p(x-y) \stackrel{?}{=} m_p(y-x) = m_p(-(x-y))$ ✓

(iv) $d_p(x,z) \stackrel{?}{\leq} d(x,y) + d(y,z)$

$$p^{-m_p(x-z)} \qquad\qquad p^{-m_p(x-y)} + p^{-m_p(y-z)}$$

Actually, $m_p(a+b) \geq \min(m_p(a), m_p(b))$. (factor out power of p)

Therefore, since $x - z = (x-y) + (y-z)$ we get

$$\underbrace{d_p(x,z) \leq \max(d_p(x,y), d_p(y,z))}_{\text{weird!}} \leq d_p(x,y) + d_p(y,z)$$

Consider: $\underbrace{0, 3}_{\frac{1}{3}}, \underbrace{\ , 12}_{\frac{1}{9}}, \underbrace{\ , 39}_{\frac{1}{27}}, \underbrace{\ , 120}_{\frac{1}{81}}, \underbrace{\ , 363}_{\frac{1}{243}}, \ldots$ convergent?

Add limit points to get the 3-adic integers: $\mathbb{Z}_3$.