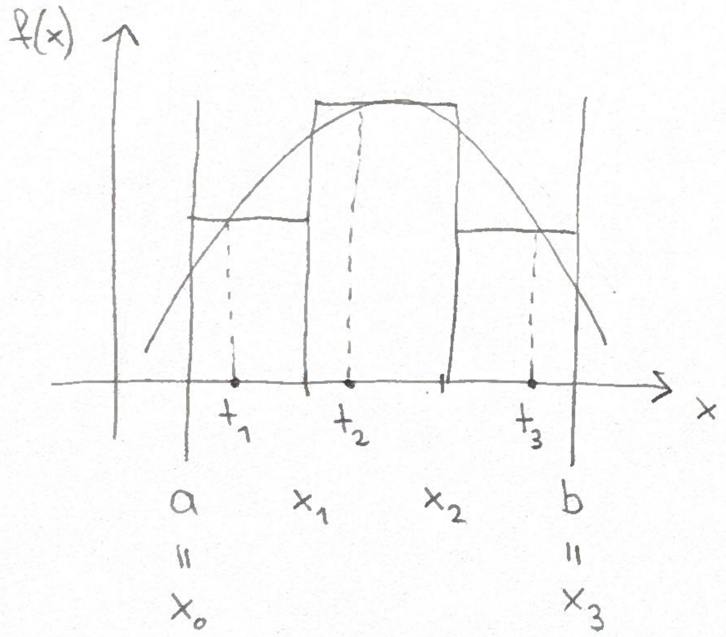
A measure of integration

(BMath Meetup, ≈ 45 minutes, 13/04/26)

- a recent seminar speaker mentioned measurable versus non-measurable sets; what does this mean?
- at my algebra lectures, Hamish often asks me to solve weird integrals / differential equations that Mike has given to him as part of a DE course;
  I don't care so much for the value of a specific integral but what is integration anyway?

$$\int_1^2 x+1 \, dx = \left(\tfrac{1}{2}x^2 + x\right)\Big|_1^2 = 4 - 1\tfrac{1}{2} = 2\tfrac{1}{2}.$$

But where does this rule come from? What if the function is more complicated?
It all goes back to Riemann integration.



$$S(P,t) := \sum_{i=0}^{n-1} f(t_i)\cdot(x_{i+1}-x_i)$$

Define $\delta(P) := \max_i |x_{i+1}-x_i|$.

Define $\int_a^b f(x)\,dx := \lim_{\delta(P)\to 0} S(P,t)$

(if it exists)

Then go through a whole lot of work to arrive at simple formulas like above. Well done, Riemann! (1868)

This definition of integration has a few flaws, some being more obvious than others.

- What if we want to integrate over something other than (intervals of) real numbers?

  Say $\mathbb{R}^2$? Partitions depend on the order of real numbers. Do we use squares of area less than δ to fill the domain?

  How about integrating over curved surfaces? How do we even measure the size of some small part of the domain?

  What if the set we want to integrate over is not even continuous but discrete?

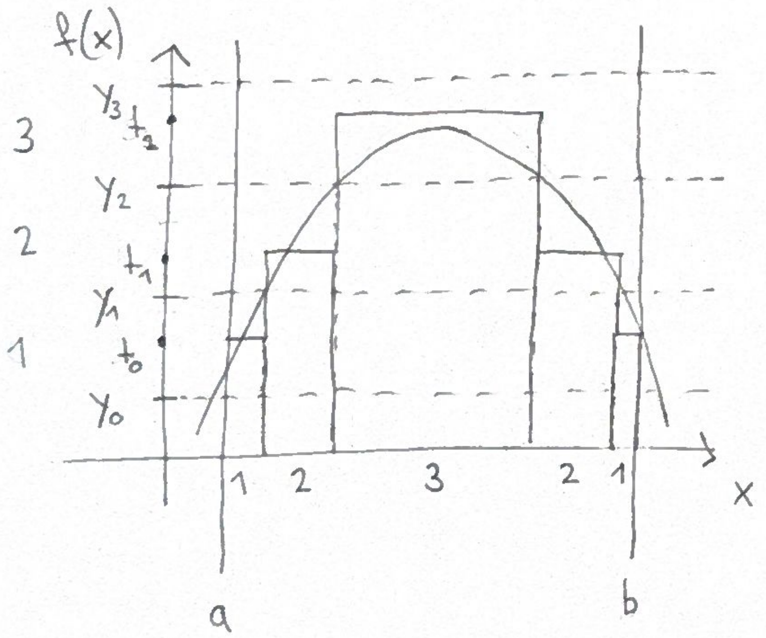
- Riemann integration struggles with discontinuous functions. Take

$$f: \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

  The Riemann integral $\int_a^b f(x)\,dx$ is not defined because every interval $[x_i, x_{i+1}]$, no matter how small, contains both rational and real numbers.

- more flaws (e.g. with limiting processes)

A <u>simple but powerful</u> idea helps to solve many of these problems.



$$S(P,t) := \sum_{i=0}^{n-1} t_i \cdot m\left(\{x \in \mathbb{R} \mid y_i \leq f(x) \leq y_{i+1}\}\right)$$

"measure" / "size" of the set of real numbers where the condition is satisfied

Define $\int_a^b f(x)\,dx := \lim\limits_{\delta(P)\to 0} S(P,t)$

(and hope for the best)

Notice a few things:

- in principal, this would allow us to integrate functions $f : X \to \mathbb{R}$ for any set X with a suitable "measure" of subsets
- in the case of the function

$$f : \mathbb{R} \longrightarrow \mathbb{R} \,, \quad x \longmapsto \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \end{cases}$$

integration would become exceedingly simple:

$$\int_a^b f(x)\, dx = 1 \cdot m\big(\mathbb{Q} \cap [a,b]\big) + 0 \cdot m\big([a,b] \setminus \mathbb{Q}\big)$$

- a lot of other problems of Riemann integration become easier, too

---

Is there a function m from subsets of $\mathbb{R}$ to $\mathbb{R}_{\geq 0}$ that <u>measures</u> size in a familiar way? We would want:

(i) $m(\emptyset) = 0$

(ii) if $X \subseteq Y$ then $m(X) \leq m(Y)$

(iii) $m([a,b]) = b-a$ whenever $b > a$

(iv) for any $t \in \mathbb{R}$ and $X \subseteq \mathbb{R}$: $m(t+X) = m(X)$

(v) for any collection of sets X_n $(n \in \mathbb{N})$ such that $X_n \cap X_m = \emptyset$ whenever $n \neq m$ we have

$$m\Big(\bigcup_n X_n\Big) = \sum_n m(X_n) \qquad \text{(possibly infinite)}$$

Unfortunately, no such function exists, e.g. due to Vitali's construction:
Consider $X := \mathbb{R} \backslash \mathbb{Q} = \{ t + \mathbb{Q} \mid t \in \mathbb{R} \}$. By the axiom of choice, there is a function $f : X \longrightarrow \mathbb{R}$ such that $f(t) \in t + \mathbb{Q}$ for all $t \in \mathbb{R}$.
By shifting values of f, assume $N := f(X) \subseteq [0,1]$. Consider $m(N)$
If $m(N) = 0$ then $m(\mathbb{R}) = m\Big(\bigcup_{t \in \mathbb{Q}} t + \mathbb{Q}\Big) = \sum_{t \in \mathbb{Q}} m(\mathbb{Q}) = 0$ by (iv) & (v), contradicting (ii). If $m(N) = c > 0$, consider the set $M := \bigcup_{t \in [0,1] \cap \mathbb{Q}} t + N \subseteq [0,2]$.
Then $2 = m([0,2]) \geq m(M) = \sum_{t \in [0,1] \cap \mathbb{Q}} m(t+N) = \sum_{t \in [0,1] \cap \mathbb{Q}} c = \infty$.

But: there is a (huge) class of well-behaved sets $\mathcal{B}$, the so called σ-algebra of Borel sets for which a measure exists.
The collection $\mathcal{B}$ contains:

- $\emptyset$
- $\mathbb{R}$
- all intervals $[a,b]$
- complements
- countable unions
- countable intersections

Thm. (Lebesgue 1902) There is a unique function $m: \mathcal{B} \to \mathbb{R}_{\geq 0}$ which satisfies (i) – (v).

This can be used to redefine integration on $\mathbb{R}$ based on the ideas described above.
The concept of "σ-algebra" and "measure" can be generalised to arbitrary sets, allowing for integration.