Uniscalar (P)-closed groups acting on trees 16/06/26

(Groups acting on non-positively curved spaces, Heidelberg, 60 minutes)

Thanks to the organiser, Bianca Marchionna!

Joint work with Ph. D. student Marcus Chijoff and co-supervisor Michal Ferov.

Willis theory of the scale and tidy subgroups for t.d.l.c. groups, such as (profinite, discrete groups), $(\mathbb{Q}_p, +)$, p-adic matrix groups, automorphism groups of trees, buildings, Kac-Moody groups.

Def. (Willis '94 +) Let G be a t.d.l.c. group and $g \in G$. The scale of g is $s(g) := \min\{[gUg^{-1} : gUg^{-1} \cap U] \mid U \leq G \text{ compact open}\} \in \mathbb{N}$.
Subgroups that minimise the index are called tidy.

(Tidy subgroups also admit a structural description which helps to identify them.)

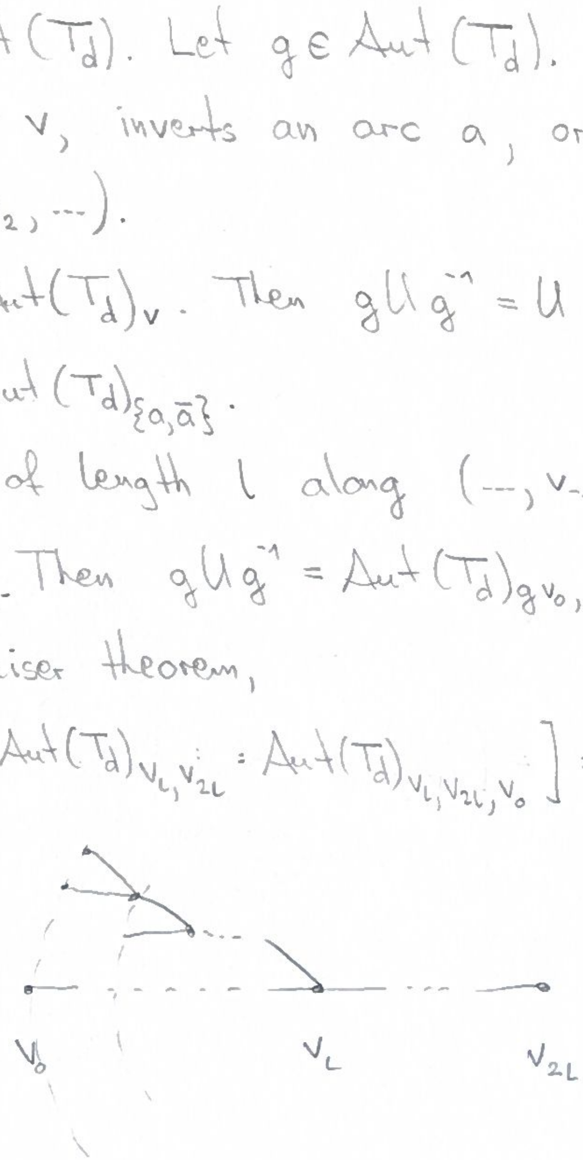
If $s(g) = 1$ for all $g \in G$ then G is uniscalar.

Ex. The scale function of $\mathrm{Aut}(T_d)$. Let $g \in \mathrm{Aut}(T_d)$.
Then g either fixes a vertex v, inverts an arc a, or translates along a line $(\dots, v_{-2}, v_{-1}, v_0, v_1, v_2, \dots)$.

- if $gv = v$, take $U := \mathrm{Aut}(T_d)_v$. Then $gUg^{-1} = U$, so $s(g) = 1$.
- if $ga = \bar{a}$, take $U := \mathrm{Aut}(T_d)_{\{a, \bar{a}\}}$.
- if g is a translation of length l along $(\dots, v_{-1}, v_0, v_1, \dots)$, consider $U := \mathrm{Aut}(T_d)_{v_0, v_l}$. Then $gUg^{-1} = \mathrm{Aut}(T_d)_{gv_0, gv_l} = \mathrm{Aut}(T_d)_{v_l, v_{2l}}$. and, by the orbit-stabiliser theorem,

$$[gUg^{-1} : gUg^{-1} \cap U] = [\mathrm{Aut}(T_d)_{v_l, v_{2l}} : \mathrm{Aut}(T_d)_{v_l, v_{2l}, v_0}] = |\mathrm{Aut}(T_d)_{v_l, v_{2l}} \cdot v_0|$$
$$= (d-1)^l$$

(one can show that this U is minimising)



Uniscalar groups are the trivial ones as far as scale theory is concerned. What are they?

- compact (i.e. profinite) groups $(U := G)$
- discrete groups $(U := \{id\})$

$\leadsto$ Wesolek's elementary groups

- t.d.l.c. groups with a compact open normal subgroup U, e.g. $\mathbb{Z}_p \times \mathbb{Z}$
- there are (compactly generated, simple) uniscalar t.d.l.c. groups that do not have a compact open normal subgroup (combinations of Glöckner, Kepert, Willis)

Q. (Th. Weigel '23) What are some classes of t.d.l.c. groups in which being uniscalar implies the existence of a compact open normal subgroup?

Thm. (Chijoff - Ferov - T. '26) Let G be a uniscalar (P)-closed group acting on a tree with compact vertex stabilisers. Then G is either horocyclic or admits a compact open normal subgroup.

Q. Can one drop the adjective "(P)-closed" from the above?

Reid-Smith (26', arXiv '20) strikingly classified (P)-closed groups acting on trees using local action diagrams.

Def. A local action diagram is a triple $\Delta = (\Gamma, (X_a)_{a \in A\Gamma}, (G(v))_{v \in V\Gamma})$ consisting of

- a connected graph $\Gamma = (V\Gamma, A\Gamma, o, t, r)$
- pairwise disjoint sets X_a $(a \in A\Gamma)$
- permutation groups $G(v) \leq \mathrm{Sym}(X_v)$, where $X_v = \bigcup_{a \in o^{-1}(v)} X_a$, with orbits X_a

Def. Let T be a tree and $H \leq \mathrm{Aut}(T)$. The (P)-closure of H is

$$H^{(P)} := \{ g \in \mathrm{Aut}(T) \mid \forall v \in VT \ \forall S \subseteq_{\text{finite}} V(B(v,1)) \ \exists h \in H : g|_S = h|_S \}$$

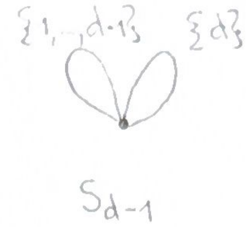
need only consider $S = V(B(v,1))$ when T is locally finite

We say H is (P)-closed if $H = H^{(P)}$.
Equivalent to Tits' (P) for closed subgroups.

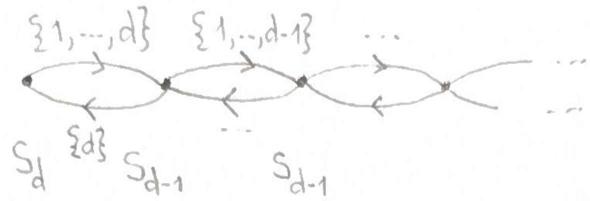
① To get a local action diagram from a group, decorate the quotient graph:

- $\mathrm{Aut}(T_d)$

- $\mathrm{Aut}(T_d)_\omega$

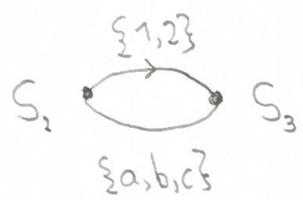


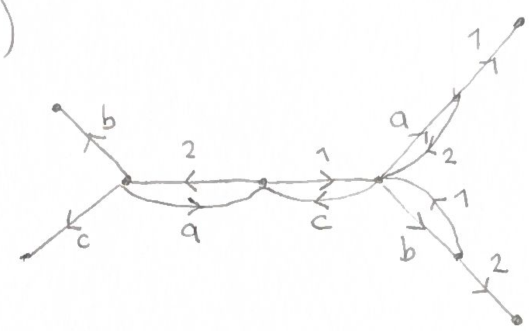
- $\mathrm{Aut}(T_d)_v$



② To get a group from a local action diagram, act on the universal cover:

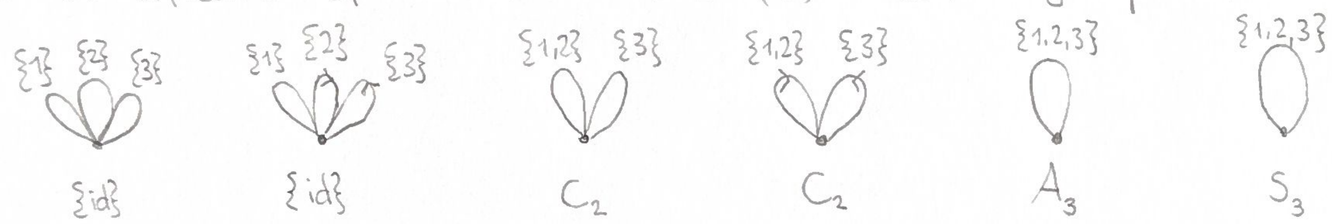
Δ $\qquad\qquad$ $(T, \pi, \mathcal{L})$





Using the coloured tree, define $U(\Delta) := \{ g \in \mathrm{Aut}_\pi(T) \mid \forall v \in VT : \sigma_{\mathcal{L}}(g,v) \in G(\pi(v)) \}$

Ex. Classification of vertex-transitive (P)-closed subgroups of $\mathrm{Aut}(T_3)$.

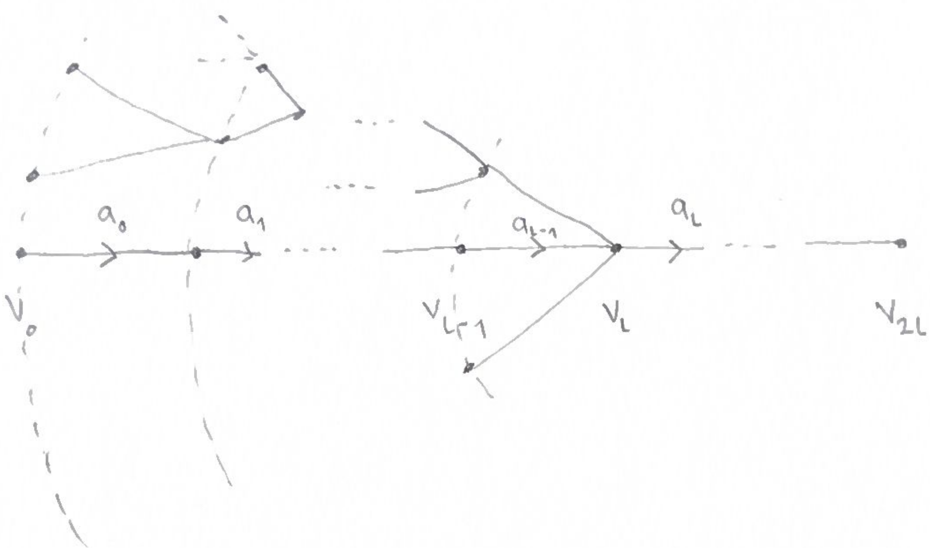


BM $\quad$ $C_2 * \mathbb{Z}$ $\quad$ BM $\quad$ $\mathrm{Aut}(T_3)_\omega$ $\quad$ BM $\quad$ BM

Properties of the group can be characterised using / determined from the local action diagram, such as

- local compactness, compact generation, top. simplicity (Reid-Smith '26)
- action type, discreteness (Chijoff - T. '24)
- translation axes, unimodularity, scale function, uniscalarity (Chijoff - Ferov - T. '26)

What does the scale function look like for translations in (P)-closed groups?

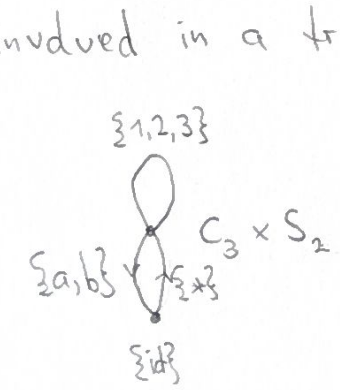


$$s(g) = \left| G(\pi(v_l))_{\mathcal{L}(a_l)} \cdot \mathcal{L}(\overline{a_{l-1}}) \right| \cdot \left| G(\pi(v_{l-1}))_{\mathcal{L}(a_{l-1})} \cdot \mathcal{L}(\overline{a_{l-2}}) \right| \cdot \cdots$$

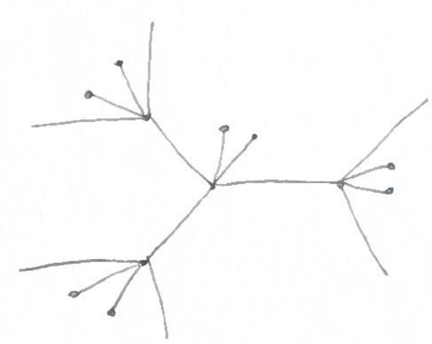
$$= \prod_{k=1}^{l} \left| G(\pi(v_k))_{\mathcal{L}(a_k)} \cdot \mathcal{L}(\overline{a_{k-1}}) \right|$$

This suggests that uniscalarity requires semiregular local actions, which would actually result in discreteness.

Something slightly less strong is true because not every colour need to be involved in a translation axis. For example:



This is a 3-regular tree T_3 with two edges attached to every vertex.
Note that T_3 is invariant.



It suffices that the local actions restricted to the invariant subtree are semiregular. The compact open normal subgroup is then given by $G_a = G_{T_3}$ for any $a \in AT_3$. The general picture is analogous.

Groups of type (Fixed Vertex), (Inversion), (Lineal) and (Horocyclic) are always uniscalar. (Focal) groups are never uniscalar.
(General) type as above.